Mathematical Olympiad Finals 2000

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1 Consider the points $O(0,0)$ and $A(0,1/2)$ on the coordinate plane. Prove that there is no finite sequence of rational points $P_1, P_2, \ldots, P_n$ in the plane such that

$$OP_1 = P_1P_2 = \ldots = P_{n-1}P_n = P_nA = 1$$

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2 Let $3n$ cards, denoted by distinct letters $a_1, a_2, \ldots, a_{3n}$, be put in line in this order from left to right. After each shuffle, the sequence $a_1, a_2, \ldots, a_{3n}$ is replaced by the sequence $a_3, a_6, \ldots, a_{3n}, a_2, a_5, \ldots,$
Is it possible to replace the sequence of cards $1, 2, \ldots, 192$ by the reverse sequence $192, 191, \ldots, 1$ by a finite number of shuffles?

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3 Given five points A, B, C, D, E in a plane, no three of which are collinear, prove the inequality

$$AB + BC + CA + DE \leq AD + AE + BD + BE + CD + CE$$

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4 Given a natural number $n \geq 3$, prove that there exists a set A_n with the following two properties:
1) A_n consists of n distinct natural numbers
2) For any $a \in A$, the remainder of the product of all elements of $A_n \backslash \{a\}$ divided by a is 1.

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5 Finitely many lines are given in a plane. We call an *intersection point* a point that belongs to at least two of the given lines, and a *good intersection point* a point that belongs to exactly two lines. Assuming there at least two intersection points, find the minimum number of good intersection points.
