Mathematical Olympiad Finals 2002

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1 Distinct points A, M, B with $AM = MB$ are given on circle (C_0) in this order. Let P be a point on the arc AB not containing M. Circle (C_1) is internally tangent to (C_0) at P and tangent to AB at Q. Prove that the product $MP \cdot MQ$ is independent of the position of P.

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2 There are $n \geq 3$ coins on a circle. Consider a coin and the two coins adjacent to it; if there are an odd number of heads among the three, we call it *good*. An operation consists of turning over all good coins simultaneously. Initially, exactly one of the n coins is a head. The operation is repeatedly performed.
(a) Prove that if n is odd, the coins will never be all-tails.
(b) For which values of n is it possible to make the coins all-tails after several operations? Find, in terms of n, the number of operations needed for this to occur.

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3 Denote by $S(n)$ the sum of decimal digits of a positive integer n. Show that there exist 2002 distinct positive integers $n_1, n_2, \cdots, n_{2002}$ such that $n_1 + S(n_1) = n_2 + S(n_2) = \cdots = n_{2002} + S(n_{2002})$.

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4 Let $n \geq 3$ be natural numbers, and let $a_1,\ a_2,\ \cdots,\ a_n,\ \ b_1,\ b_2,\ \cdots,\ b_n$ be positive numbers such that $a_1 + a_2 + \cdots + a_n = 1,\ b_1^2 + b_2^2 + \cdots + b_n^2 = 1$. Prove that $a_1(b_1 + a_2) + a_2(b_2 + a_3) + \cdots + a_n(b_n + a_1) < 1$.

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5 Let S be a set of 2002 points in the coordinate plane, no two of which have the same $x-$ or $y-$coordinate. For any two points $P, Q \in S$, consider the rectangle with one diagonal PQ and the sides parallel to the axes. Denote by W_{PQ} the number of points of S lying in the interior of this rectangle. Determine the maximum N such that, no matter how the points of S are distributed, there always exist points P, Q in S with $W_{PQ} \geq N$.
