Mathematical Olympiad Finals 2004

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1 Show that there is no positive integer n such that $2n^2-1, 3n^2-1, 6n^2-1$ are all perfect squares.

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2 Find all functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $f(xf(x)+f(y))=(f(x))^2+y$ for all $x,y\in\mathbb{R}$.

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3 Given two planes $\pi_1,\ \pi_2$ intersecting orthogonally in space. Let A, B be two distinct points on the line of intersection of π_1 and π_2, and C be the point which is on π_2 but not on π_1. Denote by P the intersection point of the bisector of $\angle BCA$ and AB, and denote S by the circumference on π_1 with a diameter AB. For an arbiterary plane π_3 which contains CP, if D, E are the intersection points of π_3 and S, then prove that CP is the bisector of $\angle DCE$.

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4 For positive real numbers a, b, c satisfying $a+b+c=1$,

Prove that we have $\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leqq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$. Note that you don't need to state for the condition
for which the equality holds.

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5 In a land any towns are connected by roads with exactly other three towns. Last year we made a trip starting from a town coming back to the town by visiting exactly one time all towns in land. This year we are plan to make a trip in the same way as last year's trip. Note that you can't take such order completely same as last year's one or trace the order only reversely. Prove that this is possible.
