Mathematical Olympiad Finals 2006

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1 Given five distinct points A, M, B, C, D in this order on the circumference of the circle O such that $MA = MB.$
Let P, Q be the intersection points of the line AC and MD, and that of the line BD and MC, respectively.
If two intersection points of the line PQ and the circumference of the circle O are $X,\ Y$, then prove that $MX = MY.$

2 Determine all integers k for which there exist infinitely the pairs of integers (a, b, c) satisfying the following equation.

$$(a^2-k)(b^2-k)=c^2-k.$$

3 Find all functions f, defined on real numbers and taking real values such that $\{f(x)\}^2+2yf(x)+f(y)=f(y+f(x))$ for all real numbers x, y.

4 Let $m,\ n$ be integers such that $2\leq m\leq n$ and let $a,\ a'$ be integers which are less than or equal to m and let $b,\ b'$ be integers which are less than or equal to n such that $(a, b)\neq (a'\ b')$. Given a town of the rectangular shaped chessboard which is made up of $m's$ road running north and south which is called Line and $n's$ road running west and east which is called Street. Denote the intersection point of the a th Line from the west and b th Street from the north by A, and a' th Line from the west and b' th Street from the north by B, including the edge for both cases.Find all pair of (m, n, a, b, a', b') such that by passing through each crossroads of the town exactly one time, you can reach the point B from the point A including in the start point and goal one.

5 For any positive real numbers $x_1,\ x_2,\ x_3,\ y_1,\ y_2,\ y_3,\ z_1,\ z_2,\ z_3,$ find the maximum value of real number A such that if

$$M=(x_1^3+x_2^3+x_3^3+1)(y_1^3+y_2^3+y_3^3+1)(z_1^3+z_2^3+z_3^3+1)$$

and

$$N=A(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3),$$

then $M\geq N$ always holds, then find the condition that the equality holds.