Mathematical Olympiad Finals 2007

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1 Let n be positive integers. Two persons play a game in which they are calling a integer m $(1 \leq m \leq n)$ alternately. Note that you may not call the number which have already said. The game is over when no one can call numbers, if the sum of the numbers that the lead have said is divisible by 3, then the lead wins, otherwise the the second move wins. Find n for which there exists the way of forestalling.

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2 Find all functions f, defined on the positive real numbers and taking real numbers such that

$$f(x)+f(y)\leq \frac{f(x+y)}{2},\ \ \frac{f(x)}{x}+\frac{f(y)}{y}\geq \frac{f(x+y)}{x+y}$$

for all $x,\ y>0.$

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3 Let Γ be the circumcircle of triangle ABC. Denote the circle which touches to the sides $AB,\ AC$ and touches to Γ internally at P by Γ_A, and the circle which touches to the sides $AB,\ BC$ and touches to Γ internally at Q by Γ_B, and the circle which touches to the sides $AC,\ BC$ and touches to Γ internally at R by Γ_C. Prove that the lines $AP,\ BQ,\ CR$ are concurrent.

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4 On a plane, call the band with width d be the set of all points which are away the distance of less than or equal to $\frac{d}{2}$ from a line.Given four points $A,\ B,\ C,\ D$ on a plane.If you take three points among them, there exists the band with width 1 containing them. Prove that there exist the band with width $\sqrt{2}$ containing all four points .

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5 For real positive numbers x, the set $A(x)$ is defined by

$$A(x)=\{[nx]\ |\ n\in\mathbb{N}\},$$

where $[r]$ denotes the greatest integer not exceeding real numbers r.Find all irrational numbers $\alpha>1$ satisfying the following condition.
Condition: If positive real number β satisfies $A(\alpha)\supset A(\beta)$, then $\frac{\beta}{\alpha}$ is integer.
