Mathematical Olympiad Finals 2008

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1 Let $P(x)$ be a polynomial with integer coefficients such that $P(n^2)=0$ for some non zero integers n. Prove that $P(a^2)\neq 1$ for all non zero rational numbers $a\neq 0$.

2 There are 2008 red cards and 2008 white cards. 2008 players sit down in circular toward the inside of the circle in situation that 2 red cards and 2 white cards from each card are delivered to each person. Each person conducts the following procedure in one turn as follows.
$(*)$ If you have more than one red card, then you will pass one red card to the left-neighbouring player.

If you have no red card, then you will pass one white card to the left -neighbouring player.
Find the maximum value of the number of turn required for the state such that all person will have one red card and one white card first.

3 Given an acute-angled triangle ABC with circumcenter O. The circle passing through two points $A,\ O$ intersects with the line AB and AC at $P,\ Q$ other than A respectively. If the lengths of the line segments $PQ,\ BC$ are equal, then find the angle $\leq 90^\circ$ that the lines PQ and BC make.

4 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)f(f(x)-y)=xf(x)-yf(y)$$

for all $x,y\in\mathbb{R}$.

5 Can it be existed postive integers n such that there are integers b and non zero integers $a_i\ (i=1,\ 2,\ \cdots n)$ for rational numbers r which satisfies $r=b+\sum_{i=1}^n \frac{1}{a_i}$?