Mathematical Olympiad Finals 2010

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1 Given an acute-angled triangle ABC such that $AB \neq AC$. Draw the perpendicular AH from A to BC. Suppose that if we take points P, Q in such a way that three points A, B, P and three points A, C, Q are collinear in this order respectively, then we have four points B, C, P, Q are concyclic and $HP = HQ$. Prove that H is the circumcenter of $\triangle APQ$.

2 Let k be positive integer and m be odd number. Prove that there exists positive integer n such that $n^n - m$ is divisible by 2^k.

3 There are 2010 islands and 2009 bridges connecting them. Suppose that any bridges are connected by one bridge or not the endpoints are connected to 2 distinct islands and we can travel a few times by crossing bridges from each island to any other islands.
Now a letter from each island was sent to some island, note that, some letter may sent to same island, then the following fact was proved that:
In case of connecting island A and island B by bridge, the habitant of island A and that of island B are mutually connected by bridge or the same island (itself).
Prove that at least one of the following statements (1) or (2) hold.

(1) There exists island for which a letter was sent to the same island.

(2) There exist 2 islands, connecting bridge, whose letter are exchanged each other.

4 Let x, y, z be positive real numbers.

Prove that

$$\frac{1+yz+zx}{(1+x+y)^2} + \frac{1+zx+xy}{(1+y+z)^2} + \frac{1+xy+yz}{(1+z+x)^2} \geq 1$$

5 Given a convex 2010 polygonal whose any 3 diagonals have no intersection points except vertices. Consider *closed broken lines* which have 2010 diagonals (not including edges) and they pass through each vertex exactly one time. Find the possible maximum value of the number of self-crossing. Note that we call *closed broken lines* such that broken line $P_1P_2 \cdots P_nP_{n+1}$ has the property $P_1 = P_{n+1}$.