Mathematical Olympiad Finals 2012

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–	February 11th
1	Given a triangle ABC, the tangent of the circumcircle at A intersects with the line BC at P. Let $Q,\ R$ be the points of symmetry for P across the lines $AB,\ AC$ respectively. Prove that the line BC intersects orthogonally with the line QR.
2	Find all functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that $f(f(x+y)f(x-y))=x^2-yf(y)$ for all $x,y\in\mathbb{R}$.
3	Let p be prime. Find all possible integers n such that for all integers x, if x^n-1 is divisible by p, then x^n-1 is divisible by p^2 as well.
4	Given two triangles PAB and PCD such that $PA=PB,\ PC=PD,\ P,\ A,\ C$ and $B,\ P,\ D$ are collinear in this order respectively. The circle S_1 passing through $A,\ C$ intersects with the circle S_2 passing through $B,\ D$ at distinct points $X,\ Y$. Prove that the circumcenter of the triangle PXY is the midpoint of the centers of $S_1,\ S_2$.
5	Given is a piece at the origin of the coordinate plane. Two persons $A,\ B$ act as following. First, A marks on a lattice point, on which the piece cannot be anymore put. Then B moves the piece from the point $(x,\ y)$ to the point $(x+1,\ y)$ or $(x,\ y+1)$, a number of m times $(1\leq m\leq k)$. Note that we may not move the piece to a marked point. If A wins when B can't move any pieces, then find all possible integers k such that A will win in a finite number of moves, regardless of how B moves the piece.