Mathematical Olympiad Finals 2013

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1 Let $n,\ k$ be positive integers with $n\geq k$. There are n persons, each person belongs to exactly one of group 1, group $2,\ \cdots$, group k and more than or equal to one person belong to any groups. Show that n^2 sweets can be delivered to n persons in such way that all of the following condition are satisfied.

- At least one sweet are delivered to each person.
- a_i sweet are delivered to each person belonging to group $i\ (1\leq i\leq k)$.
- If $1\leq i<j\leq k$, then $a_i>a_j$.

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2 Find all functions $f:\mathbb{Z}\rightarrow \mathbb{R}$ such that the equality

$$f(m)+f(n)=f(mn)+f(m+n+mn)$$

holds for all $m,n\in \mathbb{Z}$.

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3 Let $n\geq 2$ be a positive integer. Find the minimum value of positive integer m for which there exist positive integers $a_1,\ a_2,\ \cdots, a_n$ such that :

- $a_1<a_2<\cdots <a_n=m$
- $\frac{a_1^2+a_2^2}{2},\ \frac{a_2^2+a_3^2}{2},\ \cdots,\ \frac{a_{n-1}^2+a_n^2}{2}$ are all square numbers.

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4 Given an acute-angled triangle ABC, let H be the orthocenter. A cirlcle passing through the points $B,\ C$ and a cirlcle with a diameter AH intersect at two distinct points $X,\ Y$. Let D be the foot of the perpendicular drawn from A to line BC, and let K be the foot of the perpendicular drawn from D to line XY. Show that $\angle{BKD}=\angle{CKD}$.

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5 Let n be a positive integer. Given are points $P_1,\ P_2,\ \cdots,\ P_{4n}$ of which any three points are not collinear. For $i=1,\ 2,\ \cdots,\ 4n$, rotating half-line P_iP_{i-1} clockwise in 90° about the pivot P_i gives half-line P_iP_{i+1}. Find the maximum value of the number of the pairs of $(i,\ j)$ such that line segments P_iP_{i+1} and P_jP_{j+1} intersect at except endpoints.
Note that : $P_0=P_{4n},\ P_{4n+1}=P_1$ and $1\leq i<j\leq 4n$.
