Dutch Mathematical Olympiad 1999

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1 Let $f:\mathbb{Z}\to\{-1,1\}$ be a function such that

$$f(mn)=f(m)f(n),\ \forall m,n\in\mathbb{Z}.$$

Show that there exists a positive integer a such that $1\le a\le 12$ and $f(a)=f(a+1)=1$.

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2 A 9×9 square consists of 81 unit squares. Some of these unit squares are painted black, and the others are painted white, such that each 2×3 rectangle and each 3×2 rectangle contain exactly 2 black unit squares and 4 white unit squares. Determine the number of black unit squares.

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3 Let $ABCD$ be a square and let ℓ be a line. Let M be the centre of the square. The diagonals of the square have length 2 and the distance from M to ℓ exceeds 1. Let A',B',C',D' be the orthogonal projections of A,B,C,D onto ℓ. Suppose that one rotates the square, such that M is invariant. The positions of A,B,C,D,A',B',C',D' change. Prove that the value of $AA'^2+BB'^2+CC'^2+DD'^2$ does not change.

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4 Consider a matrix of size 8×8, containing positive integers only. One may repeatedly transform the entries of the matrix according to the following rules:

-Multiply all entries in some row by 2.
-Subtract 1 from all entries in some column.

Prove that one can transform the given matrix into the zero matrix.

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5 Let c be a nonnegative integer, and define $a_n=n^2+c$ (for $n\ge 1$). Define d_n as the greatest common divisor of a_n and a_{n+1}.
(a) Suppose that $c=0$. Show that $d_n=1,\ \forall n\ge 1$.
(b) Suppose that $c=1$. Show that $d_n\in\{1,5\},\ \forall n\ge 1$.
(c) Show that $d_n\le 4c+1,\ \forall n\ge 1$.
