Brazil National Olympiad 1991

www.artofproblemsolving.com/community/c5104

by Johann Peter Dirichlet

1 At a party every woman dances with at least one man, and no man dances with every woman. Show that there are men M and M' and women W and W' such that M dances with W, M' dances with W', but M does not dance with W', and M' does not dance with W.

---

2 P is a point inside the triangle ABC. The line through P parallel to AB meets AC A_0 and BC at B_0. Similarly, the line through P parallel to CA meets AB at A_1 and BC at C_1, and the line through P parallel to BC meets AB at B_2 and AC at C_2. Find the point P such that $A_0B_0 = A_1B_1 = A_2C_2$.

---

3 Given $k > 0$, the sequence a_n is defined by its first two members and

$$a_{n+2} = a_{n+1} + \frac{k}{n} a_n$$

a)For which k can we write a_n as a polynomial in n?

b) For which k can we write $\frac{a_{n+1}}{a_n} = \frac{p(n)}{q(n)}$? ($p, q$ are polynomials in $\mathbb{R}[X]$).

---

4 Show that there exists $n > 2$ such that $1991 | 1999\dots91$ (with n 9's).

---

5 $P_0 = (1,0), P_1 = (1,1), P_2 = (0,1), P_3 = (0,0)$. P_{n+4} is the midpoint of P_nP_{n+1}. Q_n is the quadrilateral $P_nP_{n+1}P_{n+2}P_{n+3}$. A_n is the interior of Q_n.

Find $\cap_{n \geq 0} A_n$.
