Brazil National Olympiad 1994

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Day 1

1 The edges of a cube are labeled from 1 to 12 in an arbitrary manner. Show that it is not possible to get the sum of the edges at each vertex the same.
Show that we can get eight vertices with the same sum if one of the labels is changed to 13.

2 Given any convex polygon, show that there are three consecutive vertices such that the polygon lies inside the circle through them.

3 We are given n objects of identical appearance, but different mass, and a balance which can be used to compare any two objects (but only one object can be placed in each pan at a time). How many times must we use the balance to find the heaviest object and the lightest object?

Day 2

4 Let $a, b > 0$ be reals such that

$$a^3 = a + 1b^6 = b + 3a$$

Show that $a > b$

5 Call a super-integer an infinite sequence of decimal digits: $\ldots d_n \ldots d_2 d_1$.

(Formally speaking, it is the sequence $(d_1, d_2 d_1, d_3 d_2 d_1, \ldots)$)

Given two such super-integers $\ldots c_n \ldots c_2 c_1$ and $\ldots d_n \ldots d_2 d_1$, their product $\ldots p_n \ldots p_2 p_1$ is formed by taking $p_n \ldots p_2 p_1$ to be the last n digits of the product $c_n \ldots c_2 c_1$ and $d_n \ldots d_2 d_1$.
Can we find two non-zero super-integers with zero product?
(a zero super-integer has all its digits zero)

6 A triangle has semi-perimeter s, circumradius R and inradius r. Show that it is right-angled iff $2R = s - r$.