Brazil National Olympiad 1995

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by Johann Peter Dirichlet, Arne

Day 1

1 $ABCD$ is a quadrilateral with a circumcircle centre O and an inscribed circle centre I. The diagonals intersect at S. Show that if two of O, I, S coincide, then it must be a square.

2 Find all real-valued functions on the positive integers such that $f(x+1019) = f(x)$ for all x, and $f(xy) = f(x)f(y)$ for all x, y.

3 For any positive integer $n > 1$, let $P(n)$ denote the largest prime divisor of n. Prove that there exist infinitely many positive integers n for which

$$P(n) < P(n+1) < P(n+2).$$

Day 2

4 A regular tetrahedron has side L. What is the smallest x such that the tetrahedron can be passed through a loop of twine of length x?

5 Show that no one n-th root of a rational (for n a positive integer) can be a root of the polynomial $x^5 - x^4 - 4x^3 + 4x^2 + 2$.

6 X has n elements. F is a family of subsets of X each with three elements, such that any two of the subsets have at most one element in common. Show that there is a subset of X with at least $\sqrt{2n}$ members which does not contain any members of F.