Brazil National Olympiad 1996

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Day 1

1 Show that there exists infinite triples $(x, y, z) \in N^3$ such that $x^2 + y^2 + z^2 = 3xyz$.

2 Does there exist a set of $n > 2, n < \infty$ points in the plane such that no three are collinear and the circumcenter of any three points of the set is also in the set?

3 Let $f(n)$ be the smallest number of 1s needed to represent the positive integer n using only 1s, $+$ signs, $\times$ signs and brackets $(,)$. For example, you could represent 80 with 13 1s as follows: $(1+1+1+1+1)(1+1+1+1)(1+1+1+1)$. Show that $3\log(n) \leq \log(3)f(n) \leq 5\log(n)$ for $n > 1$.

Day 2

4 ABC is acute-angled. D s a variable point on the side BC. O_1 is the circumcenter of ABD, O_2 is the circumcenter of ACD, and O is the circumcenter of AO_1O_2. Find the locus of O.

5 There are n boys $B_1, B_2, ..., B_n$ and n girls $G_1, G_2, ..., G_n$. Each boy ranks the girls in order of preference, and each girl ranks the boys in order of preference. Show that we can arrange the boys and girls into n pairs so that we cannot find a boy and a girl who prefer each other to their partners.

For example if (B_1, G_3) and (B_4, G_7) are two of the pairs, then it must not be the case that B_4 prefers G_3 to G_7 and G_3 prefers B_4 to B_1.

6 Let p(x) be the polynomial $x^3 + 14x^2 - 2x + 1$. Let $p^n(x)$ denote $p(p^(n-1)(x))$. Show that there is an integer N such that $p^N(x) - x$ is divisible by 101 for all integers x.