Brazil National Olympiad 1997

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Day 1

1 Given $R, r > 0$. Two circles are drawn radius R, r which meet in two points. The line joining the two points is a distance D from the center of one circle and a distance d from the center of the other. What is the smallest possible value for $D + d$?

2 Let A be a set of n non-negative integers. We say it has property $\mathcal{P}$ if the set $\{x + y \mid x, y \in A\}$ has $\binom{n}{2}$ elements. We call the largest element of A minus the smallest element, the diameter of A. Let $f(n)$ be the smallest diameter of any set A with property $\mathcal{P}$. Show that $n^2 \leq 4f(n) < 4n^3$.

(If you have some amount of time, try a best estimative for $f(n)$, such that $f(p) < 2p^2$ for prime p).

3 a) Show that there are no functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $g(f(x)) = x^3$ and $f(g(x)) = x^2$ for all $x \in \mathbb{R}$.

b) Let S be the set of all real numbers greater than 1. Show that there are functions $f, g : S \to S$ satsfying the condition above.

Day 2

4 Let $V_n = \sqrt{F_n^2 + F_{n+2}^2}$, where F_n is the Fibonacci sequence ($F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$)

Show that V_n, V_{n+1}, V_{n+2} are the sides of a triangle with area $1/2$

5 Let $f(x) = x^2 - C$ where C is a rational constant.
Show that exists only finitely many rationals x such that $\{x, f(x), f(f(x)), \ldots\}$ is finite

6 f is a plane map onto itself such that points at distance 1 are always taken at point at distance 1.
Show that f preserves distances.