Brazil National Olympiad 1998

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Day 1

1 15 positive integers, all less than 1998(and no one equal to 1), are relatively prime (no pair has a common factor ¿ 1).
Show that at least one of them must be prime.

2 Let ABC be a triangle. D is the midpoint of AB, E is a point on the side BC such that $BE = 2EC$ and $\angle ADC = \angle BAE$. Find $\angle BAC$.

3 Two players play a game as follows: there $n > 1$ rounds and $d \geq 1$ is fixed. In the first round A picks a positive integer m_1, then B picks a positive integer $n_1 \neq m_1$. In round k (for $k = 2, \ldots, n$), A picks an integer m_k such that $m_{k-1} < m_k \leq m_{k-1} + d$. Then B picks an integer n_k such that $n_{k-1} < n_k \leq n_{k-1} + d$. A gets $\gcd(m_k, n_{k-1})$ points and B gets $\gcd(m_k, n_k)$ points. After n rounds, A wins if he has at least as many points as B, otherwise he loses.

For each (n, d) which player has a winning strategy?

Day 2

1 Two players play a game as follows. The first player chooses two non-zero integers A and B. The second player forms a quadratic with A, B and 1998 as coefficients (in any order). The first player wins iff the equation has two distinct rational roots. Show that the first player can always win.

2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying, for all $x \in \mathbb{N}$,

$$f(2f(x)) = x + 1998.$$

3 Two mathematicians, lost in Berlin, arrived on the corner of Barbarossa street with Martin Luther street and need to arrive on the corner of Meininger street with Martin Luther street. Unfortunately they don't know which direction to go along Martin Luther Street to reach Meininger Street nor how far it is, so they must go fowards and backwards along Martin Luther street until they arrive on the desired corner. What is the smallest value for a positive integer k so that they can be sure that if there are N blocks between Barbarossa street and Meininger street then they can arrive at their destination by walking no more than kN blocks (no matter what N turns out to be)?