Brazil National Olympiad 2000

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Day 1

1 A rectangular piece of paper has top edge AD. A line L from A to the bottom edge makes an angle x with the line AD. We want to trisect x. We take B and C on the vertical ege through A such that $AB = BC$. We then fold the paper so that C goes to a point C' on the line L and A goes to a point A' on the horizontal line through B. The fold takes B to B'. Show that AA' and AB' are the required trisectors.

2 Let $s(n)$ be the sum of all positive divisors of n, so $s(6) = 12$. We say n is almost perfect if $s(n) = 2n - 1$. Let $\mod(n, k)$ denote the residue of n modulo k (in other words, the remainder of dividing n by k). Put $t(n) = \mod(n, 1) + \mod(n, 2) + \cdots + \mod(n, n)$.

Show that n is almost perfect if and only if $t(n) = t(n - 1)$.

3 Define f on the positive integers by $f(n) = k^2 + k + 1$, where $n = 2^k(2l + 1)$ for some k, l nonnegative integers.
Find the smallest n such that $f(1) + f(2) + ... + f(n) \geq 123456$.

Day 2

4 An infinite road has traffic lights at intervals of 1500m. The lights are all synchronised and are alternately green for $\frac{3}{2}$ minutes and red for 1 minute. For which v can a car travel at a constant speed of vm/s without ever going through a red light?

5 Let X the set of all sequences $\{a_1, a_2, \dots, a_{2000}\}$, such that each of the first 1000 terms is 0, 1 or 2, and each of the remaining terms is 0 or 1. The *distance* between two members a and b of X is defined as the number of i for which a_i and b_i are different.

Find the number of functions $f : X \to X$ which preserve the distance.

6 Let it be is a wooden unit cube. We cut along every plane which is perpendicular to the segment joining two distinct vertices and bisects it. How many pieces do we get?