Brazil National Olympiad 2001

www.artofproblemsolving.com/community/c5114

by Johann Peter Dirichlet, Fermat -Euler

Day 1

1 Show that for any a, b, c positive reals,

$$(a+b)(a+c) \geq 2\sqrt{abc(a+b+c)}$$

2 Given $a_0 > 1$, the sequence $a_0, a_1, a_2, ...$ is such that for all $k > 0$, a_k is the smallest integer greater than a_{k-1} which is relatively prime to all the earlier terms in the sequence.
Find all a_0 for which all terms of the sequence are primes or prime powers.

3 ABC is a triangle E, F are points in AB, such that $AE = EF = FB$
D is a point at the line BC such that ED is perpendiculat to BC AD is perpendicular to CF.
The angle CFA is the triple of angle BDF. ($3\angle BDF = \angle CFA$)

Determine the ratio $\frac{DB}{DC}$.

Day 2

4 A calculator treats angles as radians. It initially displays 1. What is the largest value that can be achieved by pressing the buttons cos or sin a total of 2001 times? (So you might press cos five times, then sin six times and so on with a total of 2001 presses.)

5 An altitude of a convex quadrilateral is a line through the midpoint of a side perpendicular to the opposite side. Show that the four altitudes are concurrent iff the quadrilateral is cyclic.

6 A one-player game is played as follows: There is a bowl at each integer on the Ox-axis. All the bowls are initially empty, except for that at the origin, which contains $n \geq 2$ stones. A move is either

(A) to remove two stones from a bowl and place one in each of the two adjacent bowls, or

(B) to remove a stone from each of two adjacent bowls and to add one stone to the bowl immediately to their left.

Show that only a finite number of moves can be made and that the final position (when no more moves are possible) is independent of the moves made (for a given n).