Brazil National Olympiad 2003

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Day 1

1 Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer n.

2 Let S be a set with n elements. Take a positive integer k. Let $A_1, A_2, \dots, A_k$ be any distinct subsets of S. For each i take $B_i = A_i$ or $B_i = S - A_i$. Find the smallest k such that we can always choose B_i so that $\bigcup_{i=1}^k B_i = S$, no matter what the subsets A_i are.

3 $ABCD$ is a rhombus. Take points E, F, G, H on sides AB, BC, CD, DA respectively so that EF and GH are tangent to the incircle of $ABCD$. Show that EH and FG are parallel.

Day 2

1 Given a circle and a point A inside the circle, but not at its center. Find points B, C, D on the circle which maximise the area of the quadrilateral $ABCD$.

2 Let $f(x)$ be a real-valued function defined on the positive reals such that

(1) if $x < y$, then $f(x) < f(y)$,

(2) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

3 A graph G with n vertices is called *cool* if we can label each vertex with a different positive integer not greater than $\frac{n^2}{4}$ and find a set of non-negative integers D so that there is an edge between two vertices iff the difference between their labels is in D. Show that if n is sufficiently large we can always find a graph with n vertices which is not cool.