Brazil National Olympiad 2004

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Day 1 October 16th

1 Let $ABCD$ be a convex quadrilateral. Prove that the incircles of the triangles ABC, BCD, CDA and DAB have a point in common if, and only if, $ABCD$ is a rhombus.

2 Determine all values of n such that it is possible to divide a triangle in n smaller triangles such that there are not three collinear vertices and such that each vertex belongs to the same number of segments.

3 Let $x_1, x_2, ..., x_{2004}$ be a sequence of integer numbers such that $x_{k+3} = x_{k+2} + x_k x_{k+1}$, $\forall 1 \leq k \leq 2001$. Is it possible that more than half of the elements are negative?

Day 2 October 17th

4 Consider all the ways of writing exactly ten times each of the numbers $0, 1, 2, \ldots, 9$ in the squares of a 10×10 board.
Find the greatest integer n with the property that there is always a row or a column with n different numbers.

5 Consider the sequence $(a_n)_{n \in \mathbb{N}}$ with $a_0 = a_1 = a_2 = a_3 = 1$ and $a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2$. Prove that all the terms of this sequence are integer numbers.

6 Let a and b be real numbers. Define $f_{a,b}\colon R^2 \to R^2$ by $f_{a,b}(x;y) = (a - by - x^2; x)$. If $P = (x;y) \in R^2$, define $f^0_{a,b}(P) = P$ and $f^{k+1}_{a,b}(P) = f_{a,b}(f^k_{a,b}(P))$ for all nonnegative integers k.

The set $per(a;b)$ of the *periodic points* of $f_{a,b}$ is the set of points $P \in R^2$ such that $f^n_{a,b}(P) = P$ for some positive integer n.

Fix b. Prove that the set $A_b = \{a \in R \mid per(a;b) \neq \emptyset\}$ admits a minimum. Find this minimum.