Brazil National Olympiad 2005

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Day 1 October 22nd

1 A natural number is a *palindrome* when one obtains the same number when writing its digits in reverse order. For example, 481184, 131 and 2 are palindromes.

Determine all pairs (m, n) of positive integers such that $\underbrace{111\ldots1}_{m\text{ ones}} \times \underbrace{111\ldots1}_{n\text{ ones}}$ is a palindrome.

2 Determine the smallest real number C such that the inequality

$$C(x_1^{2005} + x_2^{2005} + \cdots + x_5^{2005}) \geq x_1x_2x_3x_4x_5(x_1^{125} + x_2^{125} + \cdots + x_5^{125})^{16}$$

holds for all positive real numbers x_1, x_2, x_3, x_4, x_5.

3 A square is contained in a cube when all of its points are in the faces or in the interior of the cube. Determine the biggest $\ell > 0$ such that there exists a square of side ℓ contained in a cube with edge 1.

Day 2 October 23rd

4 We have four charged batteries, four uncharged batteries and a radio which needs two charged batteries to work.

Suppose we don't know which batteries are charged and which ones are uncharged. Find the least number of attempts sufficient to make sure the radio will work. An attempt consists in putting two batteries in the radio and check if the radio works or not.

5 Let ABC be a triangle with all angles $\leq 120^\circ$. Let F be the Fermat point of triangle ABC, that is, the interior point of ABC such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each one of the three triangles BFC, CFA and AFB, draw its Euler line - that is, the line connecting its circumcenter and its centroid.

Prove that these three Euler lines pass through one common point.

Remark. The Fermat point F is also known as the **first Fermat point** or the **first Toricelli point** of triangle ABC.

Floor van Lamoen

6 Given positive integers a, c and integer b, prove that there exists a positive integer x such that

$$a^x + x \equiv b \pmod{c},$$

that is, there exists a positive integer x such that c is a divisor of $a^x + x - b$.