Brazil National Olympiad 2006

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Day 1

1 Let ABC be a triangle. The internal bisector of $\angle B$ meets AC in P and I is the incenter of ABC. Prove that if $AP + AB = CB$, then API is an isosceles triangle.

2 Let n be an integer, $n \geq 3$. Let $f(n)$ be the largest number of isosceles triangles whose vertices belong to some set of n points in the plane without three colinear points. Prove that there exists positive real constants a and b such that $an^2 < f(n) < bn^2$ for every integer n, $n \geq 3$.

3 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y) + f(x)) = 2f(x) + xy$$

for every reals x, y.

Day 2

4 A positive integer is *bold* iff it has 8 positive divisors that sum up to 3240. For example, 2006 is bold because its 8 positive divisors, 1, 2, 17, 34, 59, 118, 1003 and 2006, sum up to 3240. Find the smallest positive bold number.

5 Let P be a convex 2006-gon. The 1003 diagonals connecting opposite vertices and the 1003 lines connecting the midpoints of opposite sides are concurrent, that is, all 2006 lines have a common point. Prove that the opposite sides of P are parallel and congruent.

6 Professor Piraldo takes part in soccer matches with a lot of goals and judges a match in his own peculiar way. A match with score of m goals to n goals, $m \geq n$, is *tough* when $m \leq f(n)$, where $f(n)$ is defined by $f(0) = 0$ and, for $n \geq 1$, $f(n) = 2n - f(r) + r$, where r is the largest integer such that $r < n$ and $f(r) \leq n$.

Let $\phi = \frac{1+\sqrt{5}}{2}$. Prove that a match with score of m goals to n, $m \geq n$, is tough if $m \leq \phi n$ and is not tough if $m \geq \phi n + 1$.