Brazil National Olympiad 2007

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Day 1

1 Let $f(x) = x^2+2007x+1$. Prove that for every positive integer n, the equation $\underbrace{f(f(\ldots(f}_{n\text{ times}}(x))\ldots)) = 0$ has at least one real solution.

2 Find the number of integers c such that $-2007 \leq c \leq 2007$ and there exists an integer x such that $x^2 + c$ is a multiple of 2^{2007}.

3 Consider n points in a plane which are vertices of a convex polygon. Prove that the set of the lengths of the sides and the diagonals of the polygon has at least $\lfloor n/2 \rfloor$ elements.

Day 2

4 2007^2 unit squares are arranged forming a 2007×2007 table. Arnold and Bernold play the following game: each move by Arnold consists of taking four unit squares that forms a 2×2 square; each move by Bernold consists of taking a single unit square. They play anternatively, Arnold being the first. When Arnold is not able to perform his move, Bernold takes all the remaining unit squares. The person with more unit squares in the end is the winner.

Is it possible to Bernold to win the game, no matter how Arnold play?

5 Let $ABCD$ be a convex quadrangle, P the intersection of lines AB and CD, Q the intersection of lines AD and BC and O the intersection of diagonals AC and BD. Show that if $\angle POQ = 90^\circ$ then PO is the bisector of $\angle AOD$ and OQ is the bisector of $\angle AOB$.

6 Given real numbers $x_1 < x_2 < \ldots < x_n$ such that every real number occurs at most two times among the differences $x_j - x_i$, $1 \leq i < j \leq n$, prove that there exists at least $\lfloor n/2 \rfloor$ real numbers that occurs exactly one time among such differences.