Romania's Team Selection Test for the 1994 IMO

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– First Exam

1: Let $X_n = \{1, 2, ..., n\}$,where $n \geq 3$.
We define the measure $m(X)$ of $X \subset X_n$ as the sum of its elements.(If $|X| = 0$,then $m(X) = 0$).
A set $X \subset X_n$ is said to be even(resp. odd) if $m(X)$ is even(resp. odd).
(a)Show that the number of even sets equals the number of odd sets.
(b)Show that the sum of the measures of the even sets equals the sum of the measures of the odd sets.
(c)Compute the sum of the measures of the odd sets.

2: Let n be an odd positive integer. Prove that $((n-1)^n + 1)^2$ divides $n(n-1)^{(n-1)^n+1} + n$.

3: Prove that the sequence $a_n = 3^n - 2^n$ contains no three numbers in geometric progression.

4: Inscribe an equilateral triangle of minimum side in a given acute-angled triangle ABC (one vertex on each side).

– Second Exam

1: First of all, I'm pretty sure a_{n+1} is supposed to be $\frac{a_n}{2}$ when a_n is even, not n (and $a_n + 7$ when a_n is odd). Second of all, I think $a_1 = 1993^{1994^{1995}}$, that is, the topmost number is 1995, not 1994.

Every number > 7 is turned into one < 7 in at most 2 steps, so the minimum is ≤ 7. Since our initial term is not divisible by 7 and it's clear that the rules can't produce terms divisible by 7 when there are none, it means that the minimum is ≤ 6. $6, 5, 3$ go to 3, while $4, 2, 1$ go to 1, so there are two possibilities: the minimum is either 3 or 1.
2 is a quadratic residue modulo 7, so either all the terms of the sequence are quadratic residues modulo 7, or none are. Since the initial term is $4 \pmod 7$, it means that all terms are quadratic residues of 7, so the minimum can't be 3, meaning that it must be 1.

2: Let S_1, S_2, S_3 be spheres of radii a, b, c respectively whose centers lie on a line l. Sphere S_2 is externally tangent to S_1 and S_3, whereas S_1 and S_3 have no common points. A straight line t touches each of the spheres,
Find the sine of the angle between l and t

3: Let $a_1, a_2, ..., a_n$ be a finite sequence of 0 and 1. Under any two consecutive terms of this sequence 0 is written if the digits are equal and 1 is written otherwise. This way a new sequence

of length $n-1$ is obtained.
By repeating this procedure $n-1$ times one obtains a triangular table of 0 and 1. Find the maximum possible number of ones that can appear on this table

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4: Let be given two concentric circles of radii R and $R_1 > R$. Let quadrilateral $ABCD$ is inscribed in the smaller circle and let the rays CD, DA, AB, BC meet the larger circle at A_1, B_1, C_1, D_1 respectively.
Prove that

$$\frac{\sigma(A_1B_1C_1D_1)}{\sigma(ABCD)} \geq \frac{R_1^2}{R^2}$$

where $\sigma(P)$ denotes the area of a polygon P.

– Third Exam

1: Let p be a (positive) prime number. Suppose that real numbers $a_1, a_2, ..., a_{p+1}$ have the property that, whenever one of the numbers is deleted, the remaining numbers can be partitioned into two classes with the same arithmetic mean. Show that these numbers must be equal.

2: Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ for which $P(2) = n$.

3: Determine all integer solutions of the equation $x^n + y^n = 1994$ where $n \geq 2$

4: Find a sequence of positive integer $f(n)$, $n \in \mathbb{N}$ such that

(1) $f(n) \leq n^8$ for any $n \geq 2$, (2) for any pairwisely distinct natural numbers $a_1, a_2, \cdots, a_k$ and n, we have that

$$f(n) \neq f(a_1) + f(a_2) + \cdots + f(a_k)$$
