Brazil National Olympiad 2009

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Day 1

1 Emerald writes 2009^2 integers in a 2009×2009 table, one number in each entry of the table. She sums all the numbers in each row and in each column, obtaining 4018 sums. She notices that all sums are distinct. Is it possible that all such sums are perfect squares?

2 Let $q = 2p + 1$, $p, q > 0$ primes. Prove that there exists a multiple of q whose digits sum in decimal base is positive and at most 3.

3 There are 2009 pebbles in some points (x, y) with both coordinates integer. A operation consists in choosing a point (a, b) with four or more pebbles, removing four pebbles from (a, b) and putting one pebble in each of the points

$$(a, b-1),\ (a, b+1),\ (a-1, b),\ (a+1, b)$$

Show that after a finite number of operations each point will necessarily have at most three pebbles. Prove that the final configuration doesn't depend on the order of the operations.

Day 2

1 Prove that there exists a positive integer n_0 with the following property: for each integer $n \geq n_0$ it is possible to partition a cube into n smaller cubes.

2 Let ABC be a triangle and O its circumcenter. Lines AB and AC meet the circumcircle of OBC again in $B_1 \neq B$ and $C_1 \neq C$, respectively, lines BA and BC meet the circumcircle of OAC again in $A_2 \neq A$ and $C_2 \neq C$, respectively, and lines CA and CB meet the circumcircle of OAB in $A_3 \neq A$ and $B_3 \neq B$, respectively. Prove that lines A_2A_3, B_1B_3 and C_1C_2 have a common point.

3 Let $n > 3$ be a fixed integer and $x_1, x_2, \ldots, x_n$ be positive real numbers. Find, in terms of n, all possible real values of

$$\frac{x_1}{x_n + x_1 + x_2} + \frac{x_2}{x_1 + x_2 + x_3} + \frac{x_3}{x_2 + x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_{n-2} + x_{n-1} + x_n} + \frac{x_n}{x_{n-1} + x_n + x_1}$$