Brazil National Olympiad 2010

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by ffao

Day 1 October 16th

1 Find all functions f from the reals into the reals such that

$$f(ab) = f(a+b)$$

for all irrational a, b.

2 Let $P(x)$ be a polynomial with real coefficients. Prove that there exist positive integers n and k such that k has n digits and more than $P(n)$ positive divisors.

3 What is the biggest shadow that a cube of side length 1 can have, with the sun at its peak?
Note: "The biggest shadow of a figure with the sun at its peak" is understood to be the biggest possible area of the orthogonal projection of the figure on a plane.

Day 2 October 17th

1 Let $ABCD$ be a convex quadrilateral, and M and N the midpoints of the sides CD and AD, respectively. The lines perpendicular to AB passing through M and to BC passing through N intersect at point P. Prove that P is on the diagonal BD if and only if the diagonals AC and BD are perpendicular.

2 Determine all values of n for which there is a set S with n points, with no 3 collinear, with the following property: it is possible to paint all points of S in such a way that all angles determined by three points in S, all of the same color or of three different colors, aren't obtuse. The number of colors available is unlimited.

3 Find all pairs (a, b) of positive integers such that

$$3^a = 2b^2 + 1.$$