Brazil National Olympiad 2011

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Day 1 October 15th

1 We call a number *pal* if it doesn't have a zero digit and the sum of the squares of the digits is a perfect square. For example, 122 and 34 are pal but 304 and 12 are not pal. Prove that there exists a pal number with n digits, $n > 1$.

2 33 friends are collecting stickers for a 2011-sticker album. A distribution of stickers among the 33 friends is incomplete when there is a sticker that no friend has. Determine the least m with the following property: every distribution of stickers among the 33 friends such that, for any two friends, there are at least m stickers both don't have, is incomplete.

3 Prove that, for all convex pentagons $P_1P_2P_3P_4P_5$ with area 1, there are indices i and j (assume $P_7 = P_2$ and $P_6 = P_1$) such that:

$$\text{Area of } \triangle P_iP_{i+1}P_{i+2} \le \frac{5-\sqrt{5}}{10} \le \text{Area of } \triangle P_jP_{j+1}P_{j+2}$$

Day 2 October 16th

4 Do there exist 2011 positive integers $a_1 < a_2 < \ldots < a_{2011}$ such that $\gcd(a_i, a_j) = a_j - a_i$ for any i, j such that $1 \le i < j \le 2011$?

5 Let ABC be an acute triangle and H is orthocenter. Let D be the intersection of BH and AC and E be the intersection of CH and AB. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on BC.

6 Let $a_1, a_2, a_3, ...a_{2011}$ be nonnegative reals with sum $\frac{2011}{2}$, prove :

$|\prod_{cyc}(a_n - a_{n+1})| = |(a_1 - a_2)(a_2 - a_3)...(a_{2011} - a_1)| \le \frac{3\sqrt{3}}{16}$.