Brazil National Olympiad 2012

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Day 1

1 In a culturing of bacteria, there are two species of them: red and blue bacteria.

When two red bacteria meet, they transform into one blue bacterium.
When two blue bacteria meet, they transform into four red bacteria.
When a red and a blue bacteria meet, they transform into three red bacteria.

Find, in function of the amount of blue bacteria and the red bacteria initially in the culturing, all possible amounts of bacteria, and for every possible amount, the possible amounts of red and blue bacteria.

2 ABC is a non-isosceles triangle. T_A is the tangency point of incircle of ABC in the side BC (define T_B,T_C analogously). I_A is the ex-center relative to the side BC (define I_B,I_C analogously). X_A is the mid-point of I_BI_C (define X_B,X_C analogously).

Show that X_AT_A,X_BT_B,X_CT_C meet in a common point, colinear with the incenter and circumcenter of ABC.

3 Find the least non-negative integer n such that exists a non-negative integer k such that the last 2012 decimal digits of n^k are all 1's.

Day 2

4 There exists some integers $n, a_1, a_2, \ldots, a_{2012}$ such that

$$n^2 = \sum_{1 \le i \le 2012} {a_i}^{p_i}$$

where p_i is the i-th prime ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$) and $a_i > 1$ for all i?

5 In how many ways we can paint a $N \times N$ chessboard using 4 colours such that squares with a common side are painted with distinct colors and every 2×2 square (formed with 4 squares in consecutive lines and columns) is painted with the four colors?

6 Find all surjective functions $f\colon (0, +\infty) \to (0, +\infty)$ such that $2xf(f(x)) = f(x)(x + f(f(x)))$ for all $x > 0$.