Brazil National Olympiad 2013

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by proglote

Day 1 October 19th

1 Let Γ be a circle and A a point outside Γ. The tangent lines to Γ through A touch Γ at B and C. Let M be the midpoint of AB. The segment MC meets Γ again at D and the line AD meets Γ again at E. Given that $AB = a$, $BC = b$, compute CE in terms of a and b.

2 Arnaldo and Bernaldo play the following game: given a fixed finite set of positive integers A known by both players, Arnaldo picks a number $a \in A$ but doesn't tell it to anyone. Bernaldo thens pick an arbitrary positive integer b (not necessarily in A). Then Arnaldo tells the number of divisors of ab. Show that Bernaldo can choose b in a way that he can find out the number a chosen by Arnaldo.

3 Find all injective functions $f\colon \mathbb{R}^* \to \mathbb{R}^*$ from the non-zero reals to the non-zero reals, such that

$$f(x+y)\left(f(x)+f(y)\right) = f(xy)$$

for all non-zero reals x, y such that $x + y \neq 0$.

Day 2 October 20th

4 Find the largest n for which there exists a sequence $(a_0, a_1, \ldots, a_n)$ of non-zero digits such that, for each k, $1 \le k \le n$, the k-digit number $\overline{a_{k-1}a_{k-2}\ldots a_0} = a_{k-1}10^{k-1} + a_{k-2}10^{k-2} + \cdots + a_0$ divides the $(k+1)$-digit number $\overline{a_k a_{k-1} a_{k-2} \ldots a_0}$.

P.S.: This is basically the same problem as http://www.artofproblemsolving.com/Forum/viewtopic.php?f=

5 Let x be an irrational number between 0 and 1 and $x = 0.a_1a_2a_3\cdots$ its decimal representation. For each $k \ge 1$, let $p(k)$ denote the number of distinct sequences $a_{j+1}a_{j+2}\cdots a_{j+k}$ of k consecutive digits in the decimal representation of x. Prove that $p(k) \ge k+1$ for every positive integer k.

6 The incircle of triangle ABC touches sides BC, CA and AB at points D, E and F, respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of triangle ABC.