**Brazil National Olympiad 2014**

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**Day 1**

**1** Let $ABCD$ be a convex quadrilateral. Diagonals $AC$ and $BD$ meet at point $P$. The inradii of triangles $ABP$, $BCP$, $CDP$ and $DAP$ are equal. Prove that $ABCD$ is a rhombus.

**2** Find all integers $n, n > 1$, with the following property: for all $k$, $0 \le k < n$, there exists a multiple of $n$ whose digits sum leaves a remainder of $k$ when divided by $n$.

**3** Let $N$ be an integer, $N > 2$. Arnold and Bernold play the following game: there are initially $N$ tokens on a pile. Arnold plays first and removes $k$ tokens from the pile, $1 \le k < N$. Then Bernold removes $m$ tokens from the pile, $1 \le m \le 2k$ and so on, that is, each player, on its turn, removes a number of tokens from the pile that is between $1$ and twice the number of tokens his opponent took last. The player that removes the last token wins.

For each value of $N$, find which player has a winning strategy and describe it.

**Day 2**

**4** The infinite sequence $P_0(x), P_1(x), P_2(x), \ldots, P_n(x), \ldots$ is defined as

$$P_0(x) = x, \quad P_n(x) = P_{n-1}(x-1) \cdot P_{n-1}(x+1), \quad n \ge 1.$$

Find the largest $k$ such that $P_{2014}(x)$ is divisible by $x^k$.

**5** There is an integer in each cell of a $2m \times 2n$ table. We define the following operation: choose three cells forming an L-tromino (namely, a cell $C$ and two other cells sharing a side with $C$, one being horizontal and the other being vertical) and sum $1$ to each integer in the three chosen cells. Find a necessary and sufficient condition, in terms of $m$, $n$ and the initial numbers on the table, for which there exists a sequence of operations that makes all the numbers on the table equal.

**6** Let $ABC$ be a triangle with incenter $I$ and incircle $\omega$. Circle $\omega_A$ is externally tangent to $\omega$ and tangent to sides $AB$ and $AC$ at $A_1$ and $A_2$, respectively. Let $r_A$ be the line $A_1A_2$. Define $r_B$ and $r_C$ in a similar fashion. Lines $r_A$, $r_B$ and $r_C$ determine a triangle $XYZ$. Prove that the incenter of $XYZ$, the circumcenter of $XYZ$ and $I$ are collinear.