Brazil Undergrad MO 2005

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Day 1 October 22nd

1 Determine the number of possible values for the determinant of A, given that A is a $n \times n$ matrix with real entries such that $A^3 - A^2 - 3A + 2I = 0$, where I is the identity and 0 is the all-zero matrix.

2 Let f and g be two continuous, distinct functions from $[0,1] \to (0,+\infty)$ such that
$\int_0^1 f(x)dx = \int_0^1 g(x)dx$

Let
$y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)} dx$, for $n \geq 0$, natural.

Prove that (y_n) is an increasing and divergent sequence.

3 Let $v_1, v_2, \dots, v_n$ vectors in $\mathbb{R}^2$ such that $|v_i| \leq 1$ for $1 \leq i \leq n$ and $\sum_{i=1}^n v_i = 0$. Prove that there exists a permutation σ of $(1, 2, \dots, n)$ such that $\left|\sum_{j=1}^k v_{\sigma(j)}\right| \leq \sqrt{5}$ for every k, $1 \leq k \leq n$.

Remark: If $v = (x, y) \in \mathbb{R}^2$, $|v| = \sqrt{x^2 + y^2}$.

Day 2 October 23rd

4 Let $a_{n+1} = a_n + \frac{1}{a_n^{2005}}$ and $a_1 = 1$. Show that $\sum_{n=1}^\infty \frac{1}{na_n}$ converge.

5 Prove that

$$\sum_{n=1}^\infty \frac{1}{n^n} = \int_0^1 x^{-x}\, dx.$$

6 Prove that for any natural numbers $0 \leq i_1 < i_2 < \cdots < i_k$ and $0 \leq j_1 < j_2 < \cdots < j_k$, the matrix $A = (a_{rs})_{1 \leq r,s \leq k}$, $a_{rs} = \binom{i_r + j_s}{i_r} = \frac{(i_r+j_s)!}{i_r!\, j_s!}$ $(1 \leq r, s \leq k)$ is nonsingular.