Baltic Way 1992

www.artofproblemsolving.com/community/c5131

by WakeUp, gwen01

1 Let p, q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least 3 natural numbers greater than 1 (not necessarily different).

---

2 Denote by $d(n)$ the number of all positive divisors of a natural number n (including 1 and n). Prove that there are infinitely many n, such that $n/d(n)$ is an integer.

---

3 Find an infinite non-constant arithmetic progression of natural numbers such that each term is neither a sum of two squares, nor a sum of two cubes (of natural numbers).

---

4 Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers?

---

5 It is given that $a^2+b^2+(a+b)^2=c^2+d^2+(c+d)^2$. Prove that $a^4+b^4+(a+b)^4=c^4+d^4+(c+d)^4$.

---

6 Prove that the product of the 99 numbers $\frac{k^3-1}{k^3+1}, k=2,3,\dots,100$ is greater than $2/3$.

---

7 Let $a=\sqrt[1992]{1992}$. Which number is greater

$$\underbrace{a^{a^{a^{\dots^{a}}}}}_{1992} \quad \text{or} \quad 1992?$$

---

8 Find all integers satisfying the equation $2^x\cdot(4-x)=2x+4$.

---

9 A polynomial $f(x)=x^3+ax^2+bx+c$ is such that $b<0$ and $ab=9c$. Prove that the polynomial f has three different real roots.

---

10 Find all fourth degree polynomial $p(x)$ such that the following four conditions are satisfied:
(i) $p(x)=p(-x)$ for all x,
(ii) $p(x)\geq 0$ for all x,
(iii) $p(0)=1$
(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1-x_2|=2$.

---

11 Let Q^+ denote the set of positive rational numbers. Show that there exists one and only one function $f: Q^+\to Q^+$ satisfying the following conditions:
(i) If $0<q<1/2$ then $f(q)=1+f(q/(1-2q))$,

(ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q-1)$,
(iii) $f(q) \cdot f(1/q) = 1$ for all $q \in Q^+$.

---

12 Let N denote the set of natural numbers. Let $\phi : N \to N$ be a bijective function and assume that there exists a finite limit

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = L.$$

What are the possible values of L?

---

13 Prove that for any positive $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n}(x_i + y_i)^2}$$

holds.

---

14 There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from another one. Prove that there is a town such that all remaining towns can be reached from it.

---

15 Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accomodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

---

16 All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

---

17 Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that the diagonal AC is a diameter of the circle, while the other diagonal BD is as long as AB. The diagonals intersect at P. It is known that the length of PC is $2/5$. How long is the side CD?

---

18 Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

---

19 Let C be a circle in plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2 touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

---

20 Let $a \leq b \leq c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that

$$p(p-c) = (p-a)(p-b) = S,$$

where S is the area of the triangle.
