Baltic Way 2000

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by April, WakeUp

1 Let K be a point inside the triangle ABC. Let M and N be points such that M and K are on opposite sides of the line AB, and N and K are on opposite sides of the line BC. Assume that $\angle MAB = \angle MBA = \angle NBC = \angle NCB = \angle KAC = \angle KCA$. Show that $MBNK$ is a parallelogram.

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2 Given an isosceles triangle ABC with $\angle A = 90^\circ$. Let M be the midpoint of AB. The line passing through A and perpendicular to CM intersects the side BC at P. Prove that $\angle AMC = \angle BMP$.

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3 Given a triangle ABC with $\angle A = 90^\circ$ and $AB \neq AC$. The points D, E, F lie on the sides BC, CA, AB, respectively, in such a way that $AFDE$ is a square. Prove that the line BC, the line FE and the line tangent at the point A to the circumcircle of the triangle ABC intersect in one point.

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4 Given a triangle ABC with $\angle A = 120^\circ$. The points K and L lie on the sides AB and AC, respectively. Let BKP and CLQ be equilateral triangles constructed outside the triangle ABC. Prove that $PQ \geq \frac{\sqrt{3}}{2}(AB + AC)$.

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5 Let ABC be a triangle such that

$$\frac{BC}{AB - BC} = \frac{AB + BC}{AC}$$

Determine the ratio $\angle A : \angle C$.

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6 Fredek runs a private hotel. He claims that whenever $n \geq 3$ guests visit the hotel, it is possible to select two guests that have equally many acquaintances among the other guests, and that also have a common acquaintance or a common unknown among the guests. For which values of n is Fredek right? (Acquaintance is a symmetric relation.)

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7 In a 40×50 array of control buttons, each button has two states: on and off . By touching a button, its state and the states of all buttons in the same row and in the same column are switched. Prove that the array of control buttons may be altered from the all-off state to the all-on state by touching buttons successively, and determine the least number of touches needed to do so.

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8 Fourteen friends met at a party. One of them, Fredek, wanted to go to bed early. He said goodbye to 10 of his friends, forgot about the remaining 3, and went to bed. After a while he returned to the party, said goodbye to 10 of his friends (not necessarily the same as before), and went to bed. Later Fredek came back a number of times, each time saying goodbye to exactly 10 of his friends, and then went back to bed. As soon as he had said goodbye to each of his friends at least once, he did not come back again. In the morning Fredek realized that he had said goodbye a di fferent number of times to each of his thirteen friends! What is the smallest possible number of times that Fredek returned to the party?

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9 There is a frog jumping on a $2k \times 2k$ chessboard, composed of unit squares. The frog's jumps are $\sqrt{1+k^2}$ long and they carry the frog from the center of a square to the center of another square. Some m squares of the board are marked with an $\times$, and all the squares into which the frog can jump from an $\times$'d square (whether they carry an $\times$ or not) are marked with an $\circ$. There are n $\circ$'d squares. Prove that $n \geq m$.

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10 Two positive integers are written on the blackboard. Initially, one of them is 2000 and the other is smaller than 2000. If the arithmetic mean m of the two numbers on the blackboard is an integer, the following operation is allowed: one of the two numbers is erased and replaced by m. Prove that this operation cannot be performed more than ten times. Give an example where the operation is performed ten times.

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11 A sequence of positive integers $a_1, a_2, \ldots$ is such that for each m and n the following holds: if m is a divisor of n and $m < n$, then a_m is a divisor of a_n and $a_m < a_n$. Find the least possible value of a_{2000}.

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12 Let $x_1, x_2, \ldots x_n$ be positive integers such that no one of them is an initial fragment of any other (for example, 12 is an initial fragment of $\underline{12}, \underline{12}5$ and $\underline{12}405$). Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} < 3.$$

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13 Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression of integers such that $i | a_i$ for $i = 1, 2, \ldots, n-1$ and $n \nmid a_n$. Prove that n is a prime power.

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14 Find all positive integers n such that n is equal to 100 times the number of positive divisors of n.

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15 Let n be a positive integer not divisible by 2 or 3. Prove that for all integers k, the number $(k+1)^n - k^n - 1$ is divisible by $k^2 + k + 1$.

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16 Prove that for all positive real numbers a, b, c we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

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17 Find all real solutions to the following system of equations:

$$\begin{cases} x+y+z+t=5 \\ xy+yz+zt+tx=4 \\ xyz+yzt+ztx+txy=3 \\ xyzt=-1 \end{cases}$$

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18 Determine all positive real numbers x and y satisfying the equation

$$x+y+\frac{1}{x}+\frac{1}{y}+4=2\cdot(\sqrt{2x+1}+\sqrt{2y+1})$$

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19 Let $t \geq \frac{1}{2}$ be a real number and n a positive integer. Prove that

$$t^{2n} \geq (t-1)^{2n}+(2t-1)^n$$

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20 For every positive integer n, let

$$x_n=\frac{(2n+1)(2n+3)\cdots(4n-1)(4n+1)}{(2n)(2n+2)\cdots(4n-2)(4n)}$$

Prove that $\frac{1}{4n}<x_n-\sqrt{2}<\frac{2}{n}$.
