Baltic Way 2005

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1 Let a_0 be a positive integer. Define the sequence $\{a_n\}_{n\geq 0}$ as follows: if

$$a_n = \sum_{i=0}^{j} c_i 10^i$$

where $c_i \in \{0, 1, 2, \cdots, 9\}$, then

$$a_{n+1} = c_0^{2005} + c_1^{2005} + \cdots + c_j^{2005}.$$

Is it possible to choose a_0 such that all terms in the sequence are distinct?

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2 Let α, β and γ be three acute angles such that $\sin\alpha + \sin\beta + \sin\gamma = 1$. Show that

$$\tan^2\alpha + \tan^2\beta + \tan^2\gamma \geq \frac{3}{8}.$$

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3 Consider the sequence $\{a_k\}_{k\geq 1}$ defined by $a_1 = 1$, $a_2 = \frac{1}{2}$ and

$$a_{k+2} = a_k + \frac{1}{2}a_{k+1} + \frac{1}{4a_k a_{k+1}} \text{ for } k \geq 1.$$

Prove that

$$\frac{1}{a_1 a_3} + \frac{1}{a_2 a_4} + \frac{1}{a_3 a_5} + \cdots + \frac{1}{a_{98} a_{100}} < 4.$$

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4 Find three different polynomials $P(x)$ with real coefficients such that $P\left(x^2+1\right) = P(x)^2 + 1$ for all real x.

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5 Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \leq 1$$

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6 Let N and K be positive integers satisfying $1 \leq K \leq N$. A deck of N different playing cards is shuffled by repeating the operation of reversing the order of K topmost cards and moving these to the bottom of the deck. Prove that the deck will be back in its initial order after a number of operations not greater than $(2N/K)^2$.

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7 A rectangular array has n rows and 6 columns, where $n \geq 2$. In each cell there is written either 0 or 1. All rows in the array are different from each other. For each two rows $(x_1, x_2, x_3, x_4, x_5, x_6)$ and $(y_1, y_2, y_3, y_4, y_5, y_6)$, the row $(x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_6y_6)$ can be found in the array as well. Prove that there is a column in which at least half of the entries are zeros.

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8 Consider a 25×25 grid of unit squares. Draw with a red pen contours of squares of any size on the grid. What is the minimal number of squares we must draw in order to colour all the lines of the grid?

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9 A rectangle is divided into 200×3 unit squares. Prove that the number of ways of splitting this rectangle into rectangles of size 1×2 is divisible by 3.

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10 Let $m = 30030$ and let M be the set of its positive divisors which have exactly 2 prime factors. Determine the smallest positive integer n with the following property: for any choice of n numbers from M, there exist 3 numbers a, b, c among them satisfying $abc = m$.

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11 Let the points D and E lie on the sides BC and AC, respectively, of the triangle ABC, satisfying $BD = AE$. The line joining the circumcentres of the triangles ADC and BEC meets the lines AC and BC at K and L, respectively. Prove that $KC = LC$.

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12 Let $ABCD$ be a convex quadrilateral such that $BC = AD$. Let M and N be the midpoints of AB and CD, respectively. The lines AD and BC meet the line MN at P and Q, respectively. Prove that $CQ = DP$.

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13 What the smallest number of circles of radius $\sqrt{2}$ that are needed to cover a rectangle (a) of size 6×3? (b) of size 5×3?

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14 Let the medians of the triangle ABC meet at G. Let D and E be different points on the line BC such that $DC = CE = AB$, and let P and Q be points on the segments BD and BE, respectively, such that $2BP = PD$ and $2BQ = QE$. Determine $\angle PGQ$.

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15 Let the lines e and f be perpendicular and intersect each other at H. Let A and B lie on e and C and D lie on f, such that all five points A, B, C, D and H are distinct. Let the lines b and d pass through B and D respectively, perpendicularly to AC; let the lines a and c pass through A and C respectively, perpendicularly to BD. Let a and b intersect at X and c and d intersect at Y. Prove that XY passes through H.

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16 Let n be a positive integer, let p be prime and let q be a divisor of $(n+1)^p - n^p$. Show that p divides $q - 1$.

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17 A sequence $(x_n)_{n\geq 0}$ is defined as follows: $x_0 = a, x_1 = 2$ and $x_n = 2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1$ for all $n > 1$. Find all integers a such that $2x_{3n} - 1$ is a perfect square for all $n \geq 1$.

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18 Let x and y be positive integers and assume that $z = \frac{4xy}{x+y}$ is an odd integer. Prove that at least one divisor of z can be expressed in the form $4n - 1$ where n is a positive integer.

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19 Is it possible to find 2005 different positive square numbers such that their sum is also a square number ?

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20 Find all positive integers $n = p_1 p_2 \cdots p_k$ which divide $(p_1+1)(p_2+1)\cdots(p_k+1)$ where $p_1 p_2 \cdots p_k$ is the factorization of n into prime factors (not necessarily all distinct).
