Baltic Way 2011

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by WakeUp

1 The real numbers $x_1, \ldots, x_{2011}$ satisfy

$$x_1 + x_2 = 2x_1',\ x_2 + x_3 = 2x_2',\ \ldots,\ x_{2011} + x_1 = 2x_{2011}'$$

where $x_1', x_2', \ldots, x_{2011}'$ is a permutation of $x_1, x_2, \ldots, x_{2011}$. Prove that $x_1 = x_2 = \ldots = x_{2011}$.

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2 Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that for all integers x and y, the following holds:

$$f(f(x) - y) = f(y) - f(f(x)).$$

Show that f is bounded.

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3 A sequence $a_1, a_2, a_3, \ldots$ of non-negative integers is such that a_{n+1} is the last digit of $a_n^n + a_{n-1}$ for all $n > 2$. Is it always true that for some n_0 the sequence $a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots$ is periodic?

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4 Let a, b, c, d be non-negative reals such that $a + b + c + d = 4$. Prove the inequality

$$\frac{a}{a^3+8} + \frac{b}{b^3+8} + \frac{c}{c^3+8} + \frac{d}{d^3+8} \le \frac{4}{9}$$

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5 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(f(x)) = x^2 - x + 1$$

for all real numbers x. Determine $f(0)$.

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6 Let n be a positive integer. Prove that the number of lines which go through the origin and precisely one other point with integer coordinates $(x, y), 0 \le x, y \le n$, is at least $\frac{n^2}{4}$.

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7 Let T denote the 15-element set $\{10a + b : a, b \in \mathbb{Z}, 1 \le a < b \le 6\}$. Let S be a subset of T in which all six digits $1, 2, \ldots, 6$ appear and in which no three elements together use all these six digits. Determine the largest possible size of S.

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8 In Greifswald there are three schools called A, B and C, each of which is attended by at least one student. Among any three students, one from A, one from B and one from C, there are two knowing each other and two not knowing each other. Prove that at least one of the following holds:

-Some student from A knows all students from B.
-Some student from B knows all students from C.
- Some student from C knows all students from A.

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9 Given a rectangular grid, split into $m \times n$ squares, a colouring of the squares in two colours (black and white) is called valid if it satisfies the following conditions:

-All squares touching the border of the grid are coloured black.
-No four squares forming a 2×2 square are coloured in the same colour.
-No four squares forming a 2×2 square are coloured in such a way that only diagonally touching
squares have the same colour.
Which grid sizes $m \times n$ (with $m, n \geq 3$) have a valid colouring?

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10 Two persons play the following game with integers. The initial number is 2011^{2011}. The players move in turns. Each move consists of subtraction of an integer between 1 and 2010 inclusive, or division by 2011, rounding down to the closest integer when necessary. The player who first obtains a non-positive integer wins. Which player has a winning strategy?

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11 Let AB and CD be two diameters of the circle C. For an arbitrary point P on C, let R and S be the feet of the perpendiculars from P to AB and CD, respectively. Show that the length of RS is independent of the choice of P.

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12 Let P be a point inside a square $ABCD$ such that $PA : PB : PC$ is $1 : 2 : 3$. Determine the angle $\angle BPA$.

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13 Let E be an interior point of the convex quadrilateral $ABCD$. Construct triangles $\triangle ABF, \triangle BCG, \triangle CDH$ and $\triangle DAI$ on the outside of the quadrilateral such that the similarities $\triangle ABF \sim \triangle DCE, \triangle BCG \sim \triangle ADE, \triangle CDH \sim \triangle BAE$ and $\triangle DAI \sim \triangle CBE$ hold. Let P, Q, R and S be the projections of E on the lines AB, BC, CD and DA, respectively. Prove that if the quadrilateral $PQRS$ is cyclic, then

$$EF \cdot CD = EG \cdot DA = EH \cdot AB = EI \cdot BC.$$

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14 The incircle of a triangle ABC touches the sides BC, CA, AB at D, E, F, respectively. Let G be a point on the incircle such that FG is a diameter. The lines EG and FD intersect at H. Prove that $CH \parallel AB$.

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15 Let $ABCD$ be a convex quadrilateral such that $\angle ADB = \angle BDC$. Suppose that a point E on the side AD satisfies the equality

$$AE \cdot ED + BE^2 = CD \cdot AE.$$

Show that $\angle EBA = \angle DCB$.

16 Let a be any integer. Define the sequence $x_0, x_1, \ldots$ by $x_0 = a$, $x_1 = 3$, and for all $n > 1$

$$x_n = 2x_{n-1} - 4x_{n-2} + 3.$$

Determine the largest integer k_a for which there exists a prime p such that p^{k_a} divides $x_{2011} - 1$.

17 Determine all positive integers d such that whenever d divides a positive integer n, d will also divide any integer obtained by rearranging the digits of n.

18 Determine all pairs (p, q) of primes for which both $p^2 + q^3$ and $q^2 + p^3$ are perfect squares.

19 Let $p \neq 3$ be a prime number. Show that there is a non-constant arithmetic sequence of positive integers $x_1, x_2, \ldots, x_p$ such that the product of the terms of the sequence is a cube.

20 An integer $n \geq 1$ is called balanced if it has an even number of distinct prime divisors. Prove that there exist infinitely many positive integers n such that there are exactly two balanced numbers among $n, n+1, n+2$ and $n+3$.