Baltic Way 2014

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by socrates

1 Show that
$$\cos(56°) \cdot \cos(2 \cdot 56°) \cdot \cos(2^2 \cdot 56°) \cdot ... \cdot \cos(2^{23} \cdot 56°) = \frac{1}{2^{24}}.$$

2 Let $a_0, a_1, ..., a_N$ be real numbers satisfying $a_0 = a_N = 0$ and
$$a_{i+1} - 2a_i + a_{i-1} = a_i^2$$
for $i = 1, 2, ..., N - 1$. Prove that $a_i \leq 0$ for $i = 1, 2, ..., N - 1$.

3 Positive real numbers a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove the inequality
$$\frac{1}{\sqrt{a^3 + b}} + \frac{1}{\sqrt{b^3 + c}} + \frac{1}{\sqrt{c^3 + a}} \leq \frac{3}{\sqrt{2}}.$$

4 Find all functions f defined on all real numbers and taking real values such that
$$f(f(y)) + f(x - y) = f(xf(y) - x),$$
for all real numbers x, y.

5 Given positive real numbers a, b, c, d that satisfy equalities
$$a^2 + d^2 - ad = b^2 + c^2 + bc \text{ and } a^2 + b^2 = c^2 + d^2$$
find all possible values of the expression $\frac{ab+cd}{ad+bc}$.

6 In how many ways can we paint 16 seats in a row, each red or green, in such a way that the number of consecutive seats painted in the same colour is always odd?

7 Let $p_1, p_2, ..., p_{30}$ be a permutation of the numbers $1, 2, ..., 30$. For how many permutations does the equality $\sum_{k=1}^{30} |p_k - k| = 450$ hold?

8 Albert and Betty are playing the following game. There are 100 blue balls in a red bowl and 100 red balls in a blue bowl. In each turn a player must make one of the following moves:
a) Take two red balls from the blue bowl and put them in the red bowl.
b) Take two blue balls from the red bowl and put them in the blue bowl.

c) Take two balls of different colors from one bowl and throw the balls away.
They take alternate turns and Albert starts. The player who first takes the last red ball from the blue bowl or the last blue ball from the red bowl wins.
Determine who has a winning strategy.

9   What is the least posssible number of cells that can be marked on an $n \times n$ board such that for each $m > \frac{n}{2}$ both diagonals of any $m \times m$ sub-board contain a marked cell?

10   In a country there are 100 airports. Super-Air operates direct flights between some pairs of airports (in both directions). The *traffic* of an airport is the number of airports it has a direct Super-Air connection with. A new company, Concur-Air, establishes a direct flight between two airports if and only if the sum of their traffics is at least 100. It turns out that there exists a round-trip of Concur-Air flights that lands in every airport exactly once. Show that then there also exists a round-trip of Super-Air flights that lands in every airport exactly once.

11   Let Γ be the circumcircle of an acute triangle ABC. The perpendicular to AB from C meets AB at D and Γ again at E. The bisector of angle C meets AB at F and Γ again at G. The line GD meets Γ again at H and the line HF meets Γ again at I. Prove that $AI = EB$.

12   Triangle ABC is given. Let M be the midpoint of the segment AB and T be the midpoint of the arc BC not containing A of the circumcircle of ABC. The point K inside the triangle ABC is such that $MATK$ is an isosceles trapezoid with $AT \parallel MK$. Show that $AK = KC$.

13   Let $ABCD$ be a square inscribed in a circle ω and let P be a point on the shorter arc AB of ω. Let $CP \cap BD = R$ and $DP \cap AC = S$.
Show that triangles ARB and DSR have equal areas.

14   Let $ABCD$ be a convex quadrilateral such that the line BD bisects the angle ABC. The circumcircle of triangle ABC intersects the sides AD and CD in the points P and Q, respectively. The line through D and parallel to AC intersects the lines BC and BA at the points R and S, respectively. Prove that the points P, Q, R and S lie on a common circle.

15   The sum of the angles A and C of a convex quadrilateral $ABCD$ is less than $180°$. Prove that

$$AB \cdot CD + AD \cdot BC < AC(AB + AD).$$

16   Determine whether $712! + 1$ is a prime number.

17   Do there exist pairwise distinct rational numbers x, y and z such that

$$\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} = 2014?$$

18   Let p be a prime number, and let n be a positive integer. Find the number of quadruples (a_1, a_2, a_3, a_4) with $a_i \in \{0, 1, \ldots, p^n - 1\}$ for $i = 1, 2, 3, 4$, such that

$$p^n \mid (a_1 a_2 + a_3 a_4 + 1).$$

19   Let m and n be relatively prime positive integers. Determine all possible values of

$$\gcd(2^m - 2^n, 2^{m^2 + mn + n^2} - 1).$$

20   Consider a sequence of positive integers $a_1, a_2, a_3, \ldots$ such that for $k \geq 2$ we have $a_{k+1} = \frac{a_k + a_{k-1}}{2015^i}$, where 2015^i is the maximal power of 2015 that divides $a_k + a_{k-1}$. Prove that if this sequence is periodic then its period is divisible by 3.