All-Russian Olympiad 1993

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– Grade level 9

Day 1

1 For a positive integer n, numbers $2n+1$ and $3n+1$ are both perfect squares. Is it possible for $5n+3$ to be prime?

2 Segments AB and CD of length 1 intersect at point O and angle AOC is equal to sixty degrees. Prove that $AC+BD \ge 1$.

3 Quadratic trinomial $f(x)$ is allowed to be replaced by one of the trinomials $x^2f(1+\frac{1}{x})$ or $(x-1)^2f(\frac{1}{x-1})$. With the use of these operations, is it possible to go from x^2+4x+3 to $x^2+10x+9$?

4 In a family album, there are ten photos. On each of them, three people are pictured: in the middle stands a man, to the right of him stands his brother, and to the left of him stands his son. What is the least possible total number of people pictured, if all ten of the people standing in the middle of the ten pictures are different.

Day 2

1 For integers x, y, and z, we have $(x-y)(y-z)(z-x) = x+y+z$. Prove that $27|x+y+z$.

2 A convex quadrilateral intersects a circle at points $A_1, A_2, B_1, B_2, C_1, C_2, D_1,$ and D_2. (Note that for some letter N, points N_1 and N_2 are on one side of the quadrilateral. Also, the points lie in that specific order on the circle.) Prove that if $A_1B_2 = B_1C_2 = C_1D_2 = D_1A_2$, then quadrilateral formed by these four segments is cyclic.

3 What is the maximum number of checkers it is possible to put on a $n \times n$ chessboard such that in every row and in every column there is an even number of checkers?

4 On a board, there are n equations in the form $*x^2+*x+*$. Two people play a game where they take turns. During a turn, you are aloud to change a star into a number not equal to zero. After $3n$ moves, there will be n quadratic equations. The first player is trying to make more of the equations not have real roots, while the second player is trying to do the opposite. What is the maximum number of equations that the first player can create without real roots no matter how the second player acts?

– Grade level 10

Day 1

1 The lengths of the sides of a triangle are prime numbers of centimeters. Prove that its area cannot be an integer number of square centimeters.

2 From the symmetry center of two congruent intersecting circles, two rays are drawn that intersect the circles at four non-collinear points. Prove that these points lie on one circle.

3 Quadratic trinomial $f(x)$ is allowed to be replaced by one of the trinomials $x^2f(1+\frac{1}{x})$ or $(x-1)^2f(\frac{1}{x-1})$. With the use of these operations, is it possible to go from x^2+4x+3 to $x^2+10x+9$?

4 Thirty people sit at a round table. Each of them is either smart or dumb. Each of them is asked: "Is your neighbor to the right smart or dumb?" A smart person always answers correctly, while a dumb person can answer both correctly and incorrectly. It is known that the number of dumb people does not exceed F. What is the largest possible value of F such that knowing what the answers of the people are, you can point at at least one person, knowing he is smart?

Day 2

1 For integers x, y, and z, we have $(x-y)(y-z)(z-x)=x+y+z$. Prove that $27|x+y+z$.

2 Is it true that any two rectangles of equal area can be placed in the plane such that any horizontal line intersecting at least one of them will also intersect the other, and the segments of intersection will be equal?

3 A square is divided by horizontal and vertical lines that form n^2 squares each with side 1. What is the greatest possible value of n such that it is possible to select n squares such that any rectangle with area n formed by the horizontal and vertical lines would contain at least one of the selected n squares.

4 If $\{a_k\}$ is a sequence of real numbers, call the sequence $\{a'_k\}$ defined by $a'_k=\frac{a_k+a_{k+1}}{2}$ the *average sequence* of $\{a_k\}$. Consider the sequences $\{a_k\}$; $\{a'_k\}$ - *average sequence* of $\{a_k\}$; $\{a''_k\}$ - average sequence of $\{a'_k\}$ and so on. If all these sequences consist only of integers, then $\{a_k\}$ is called *Good*. Prove that if $\{x_k\}$ is a *good* sequence, then $\{x_k^2\}$ is also *good*.

– Grade level 11

Day 1

1 For a positive integer n, numbers $2n+1$ and $3n+1$ are both perfect squares. Is it possible for $5n+3$ to be prime?

2 Two right triangles are on a plane such that their medians (from the right angles to the hypotenuses) are parallel. Prove that the angle formed by one of the legs of one of the triangles and one of the legs of the other triangle is half the measure of the angle formed by the hypotenuses.

3 Find all functions $f(x)$ with the domain of all positive real numbers, such that for any positive numbers x and y, we have $f(x^y) = f(x)^{f(y)}$.

4 Prove that there exists a positive integer n, such that if an equilateral triangle with side lengths n is split into n^2 triangles with side lengths 1 with lines parallel to its sides, then among the vertices of the small triangles it is possible to choose $1993n$ points so that no three of them are vertices of an equilateral triangle.

Day 2

1 Find all quadruples of real numbers such that each of them is equal to the product of some two other numbers in the quadruple.

2 The integers from 1 to 1993 are written in a line in some order. The following operation is performed with this line: if the first number is k then the first k numbers are rewritten in reverse order. Prove that after some finite number of these operations, the first number in the line of numbers will be 1.

3 In a tennis tournament, n players want to make 2 vs 2 matches such that each player has each of the other players as opponents exactly once. Find all possible values of n.

4 Prove that any two rectangular prisms with equal volumes can be placed in a space such that any horizontal plain that intersects one of the prisms will intersect the other forming a polygon with the same area.