All-Russian Olympiad 1998

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– Grade level 9

Day 1

1 The angle formed by the rays $y = x$ and $y = 2x$ ($x \ge 0$) cuts off two arcs from a given parabola $y = x^2 + px + q$. Prove that the projection of one arc onto the x-axis is shorter by 1 than that of the second arc.

2 A convex polygon is partitioned into parallelograms. A vertex of the polygon is called *good* if it belongs to exactly one parallelogram. Prove that there are more than two good vertices.

3 Let $S(x)$ denote the sum of the decimal digits of x. Do there exist natural numbers a, b, c such that

$$S(a+b) < 5, \quad S(b+c) < 5, \quad S(c+a) < 5, \quad S(a+b+c) > 50?$$

4 A maze is an 8×8 board with some adjacent squares separated by walls, so that any two squares can be connected by a path not meeting any wall. Given a command LEFT, RIGHT, UP, DOWN, a pawn makes a step in the corresponding direction unless it encounters a wall or an edge of the chessboard. God writes a program consisting of a finite sequence of commands and gives it to the Devil, who then constructs a maze and places the pawn on one of the squares. Can God write a program which guarantees the pawn will visit every square despite the Devil's efforts?

Day 2

5 We are given five watches which can be winded forward. What is the smallest sum of winding intervals which allows us to set them to the same time, no matter how they were set initially?

6 In triangle ABC with $AB > BC$, BM is a median and BL is an angle bisector. The line through M and parallel to AB intersects BL at point D, and the line through L and parallel to BC intersects BM at point E. Prove that ED is perpendicular to BL.

7 A jeweller makes a chain consisting of $N > 3$ numbered links. A querulous customer then asks him to change the order of the links, in such a way that the number of links the jeweller must open is maximized. What is the maximum number?

8 Two distinct positive integers a, b are written on the board. The smaller of them is erased and replaced with the number $\frac{ab}{|a-b|}$. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

– Grade level 10

Day 1

1 Two lines parallel to the x-axis cut the graph of $y = ax^3 + bx^2 + cx + d$ in points A, C, E and B, D, F respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.

2 Two polygons are given on the plane. Assume that the distance between any two vertices of the same polygon is at most 1, and that the distance between any two vertices of different polygons is at least $1/\sqrt{2}$. Prove that these two polygons have no common interior points.

By the way, can two sides of a polygon intersect?

3 In scalene $\triangle ABC$, the tangent from the foot of the bisector of $\angle A$ to the incircle of $\triangle ABC$, other than the line BC, meets the incircle at point K_a. Points K_b and K_c are analogously defined. Prove that the lines connecting K_a, K_b, K_c with the midpoints of BC, CA, AB, respectively, have a common point on the incircle.

4 Let k be a positive integer. Some of the $2k$-element subsets of a given set are marked. Suppose that for any subset of cardinality less than or equal to $(k+1)^2$ all the marked subsets contained in it (if any) have a common element. Show that all the marked subsets have a common element.

Day 2

5 Initially the numbers 19 and 98 are written on a board. Every minute, each of the two numbers is either squared or increased by 1. Is it possible to obtain two equal numbers at some time?

6 A binary operation $*$ on real numbers has the property that $(a * b) * c = a + b + c$ for all a, b, c. Prove that $a * b = a + b$.

7 Let n be an integer at least 4. In a convex n-gon, there is NO four vertices lie on a same circle. A circle is called circumscribed if it passes through 3 vertices of the n-gon and contains all other vertices. A circumscribed circle is called boundary if it passes through 3 consecutive vertices, a circumscribed circle is called inner if it passes through 3 pairwise non-consecutive points. Prove the number of boundary circles is 2 more than the number of inner circles.

8 Each square of a $(2^n-1)\times(2^n-1)$ board contains either 1 or -1. Such an arrangement is called *successful* if each number is the product of its neighbors. Find the number of successful arrangements.

– Grade level 11

Day 1

1 Two lines parallel to the x-axis cut the graph of $y=ax^3+bx^2+cx+d$ in points A, C, E and B, D, F respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.

2 Let ABC be a triangle with circumcircle w. Let D be the midpoint of arc BC that contains A. Define E and F similarly. Let the incircle of ABC touches BC, CA, AB at K, L, M respectively. Prove that DK, EL, FM are concurrent.

3 A set S of translates of an equilateral triangle is given in the plane, and any two have nonempty intersection. Prove that there exist three points such that every triangle in S contains one of these points.

4 yeah you're right,the official problem is the following one:

there are 1998 cities in Russia, each being connected (in both directions) by flights to three other cities. any city can be reached by any other city by a sequence of flights. the KGB plans to close off 200 cities, no two joined by a single flight. show that this can be done so that any open city can be reached from any other open city by a sequence of flights only passing through open cities.

we begin with some terminology. define a **trigraph** to be a connected undirected graph in which every vertex has degree at most 3. a **trivalent** vertex of such a graph is a vertex of degree 3. in this wording, the problem becomes: we have a trigraph G with 1998 vertices, all of which are trivalent. we want to remove 200 vertices, no two of which are adjacent, such that the remaining vertices stay connected.
we remove the vertices one at a time. suppose we have deleted k of the 1998 vertices, no two of which are adjacent, such the trigraph G' induced by the remaining vertices is connected. we will show that if $K<200$, we can always delete a trivalent vertex of G' such that the graph remains connected. this vertex cannot be adjacent in G to any of the other k deleted vertices, because then its degree in G' would be less than 3. hence repeating this 200 times gives us the desired set of vertices.

Lemma. let G be a trigraph such that the removal of any trivalent vertex disconnects G. then G

is planer. moreover G can be drawn in such a way that every vertex lies on the "outside" face; in other words, for any point P outside some bounded set, each vertex v of G can be joined to P by a curve which does not intersect any edges of G (except at v).

Proof. we induct on the number of trivalent vertices of G. if G does not contain any trivalent vertices, then G must be a path or a cycle and the claim is obvious. so suppose G contains $n \geq 1$ trivalent vertices and that every trigraph with fewer trivalent vertices can be drawn as described. if G is a tree the claim is obvious, so suppose G contains a cycle; let $v_1, \ldots, v_k$ $(k \geq 3)$ be a minimal cycle. let $S = \{v_1, \ldots, v_k\}$, and let $T = \{i \mid v_i \text{ is trivalent }\}$. ($T$ cannot be empty, because then no v_i would be connected to a vertex of degree 3.) for each $i \in T$, let w_i denote the third vertex which is adjacent to v_i (other than v_{i-1} and v_{i+1}), and let S_i be the set of vertices in G which can be reached from w_i without passing though S. (for $i \notin T$, let $S_i = \emptyset$.) we claim that the sets $S, S_1, \ldots, S_k$ partition the vertices of G. first, note that if v is a vertex of G not in S then there is a shortest path joining v to a vertex v_i of S; the penultimate vertex on this path must be w_i, so $v \in S_i$. now suppose that $v \in S_i \cap S_j$ for some $i \neq j$; then v_i and v_j are trivalent and there exist paths $w_i \to v$, $w_j \to v$ which do not pass though S. we will show that there exists a path from every vertex of $G - \{v_i\}$ to v which does not pass though v_i. for $k \neq i$ there is a path $v_k \to v_j \to w_j \to v$; if $w \in S_k$ for $k \neq i$, then there is a path $w \to w_k \to v_k \to v$; if $w \in S_i$, there is a path $w \to w_i \to v$. since $S \cup S_1 \cup \ldots \cup S_k = G$, we have shown that the graph obtained from G by deleting v_i is connected,a contradiction, as v_i is trivalent. therefore $S_i \cap S_j = \emptyset$ for $i \neq j$. obviously $S_i \cap S$ is empty for all i; hence $S, S_1, \ldots, S_k$ partition the vertices of G. let $G', G_1, \ldots, G_k$ be the induced subgraphs of $S, S_1, \ldots, S_k$ in G, respectively. by construction, the only edges in G which are not in one of the graphs $G', G_1, \ldots, G_k$ are the edges $v_i w_i$ for $i \in T$. now G_i is a trigraph with fewer than n trivalent vertices, since at least one of the n trivalent vertices in G is in S. hence by the inductive hypothesis, we can draw each G_i in the plane in such a way that every vertex lies on the outside face. since $v_1, \ldots, v_k$ was a minimal cycle, there are no "extra" edges between these vertices, so the graph G' is a k-cycle. now place the vertices of S at the vertices of a small regular k-gon far from all the graphs G_i; then we can draw a curve joining each pair v_i, w_i. it is east to check that this gives us a drawing of G with the desired properties. $\square$

now suppose we have removed k vertices from G, no two of which are adjacent, such that the trigraph G' induced by the remaining vertices is connected, and suppose that removing any trivalent vertex of G' disconnects the graph; we must show $k \geq 200$. by the lemma, G' is planner. we will call a dace other than the outside one a "proper face". let F be the number of proper faces of G'; since G' has $1998 - k$ vertices and $2997 - 3k$ edges,

$$F \geq 1 - (1998 - k) + (2997 - 3k) = 1000 - 2k.$$

we now show that no two proper faces can share a vertex. observe that each vertex belongs to at most as many faces as its degree; thus vertices of degree 1 lie only on the outside face. no two proper faces can intersect in a vertex of degree 2, or that vertex would not lie on the outside face, contracting the lemma. if two proper daces intersected in a trivalent vertex v, each face would give a path between two of v's neighbors, so removing v would not disconnect the graph,

by an argument similar to that of the lemma.
since each proper face contains at least 3 vertices and no two share a vertex, we have $3F \leq 1998 - k$. combining this with the previous inequality gives

$$3000 - 6k \leq 3F \leq 1998 - k$$

so $1002 \leq 5k$ and $k \geq 200$, as desired.

Day 2

5 A sequence of distinct circles $\omega_1, \omega_2, \cdots$ is inscribed in the parabola $y = x^2$ so that ω_n and ω_{n+1} are tangent for all n. If ω_1 has diameter 1 and touches the parabola at $(0,0)$, find the diameter of ω_{1998}.

6 Are there 1998 different positive integers, the product of any two being divisible by the square of their difference?

7 A tetrahedron $ABCD$ has all edges of length less than 100, and contains two nonintersecting spheres of diameter 1. Prove that it contains a sphere of diameter 1.01.

8 A figure Φ composed of unit squares has the following property: if the squares of an $m \times n$ rectangle (m, n are fixed) are filled with numbers whose sum is positive, the figure Φ can be placed within the rectangle (possibly after being rotated) so that the sum of the covered numbers is also positive. Prove that a number of such figures can be put on the $m \times n$ rectangle so that each square is covered by the same number of figures.