All-Russian Olympiad 1999

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– Grade level 9

Day 1

1 The decimal digits of a natural number A form an increasing sequence (from left to right). Find the sum of the digits of $9A$.

2 There are several cities in a country. Some pairs of the cities are connected by a two-way airline of one of the N companies, so that each company serves exactly one airline from each city, and one can travel between any two cities, possibly with transfers. During a financial crisis, $N-1$ airlines have been canceled, all from different companies. Prove that it is still possible to travel between any two cities.

3 A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

4 Initially numbers from 1 to 1000000 are all colored black. A move consists of picking one number, then change the color (black to white or white to black) of itself and all other numbers NOT coprime with the chosen number. Can all numbers become white after finite numbers of moves?

Edited by pbornsztein

Day 2

5 An equilateral triangle of side n is divided into equilateral triangles of side 1. Find the greatest possible number of unit segments with endpoints at vertices of the small triangles that can be chosen so that no three of them are sides of a single triangle.

6 Prove that for all natural numbers n,

$$\sum_{k=1}^{n^2}\left\{\sqrt{k}\right\} \le \frac{n^2-1}{2}.$$

Here, $\{x\}$ denotes the fractional part of x.

7 A circle through vertices A and B of triangle ABC meets side BC again at D. A circle through B and C meets side AB at E and the first circle again at F. Prove that if points A, E, D, C lie on a circle with center O then $\angle BFO$ is right.

8 There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts one or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

– Grade level 10

Day 1

1 There are three empty jugs on a table. Winnie the Pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the Pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the Pooh and Piglet can cooperate so as to make Rabbit lose.

2 Find all bounded sequences $(a_n)_{n=1}^\infty$ of natural numbers such that for all $n \geq 3$,

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}.$$

3 The incircle of $\triangle ABC$ touch AB,BC,CA at K,L,M. The common external tangents to the incircles of $\triangle AMK$,$\triangle BKL$,$\triangle CLM$, distinct from the sides of $\triangle ABC$, are drawn. Show that these three lines are concurrent.

4 A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

Day 2

5 The sum of the (decimal) digits of a natural number n equals 100, and the sum of digits of $44n$ equals 800. Determine the sum of digits of $3n$.

6 In triangle ABC, a circle passes through A and B and is tangent to BC. Also, a circle that passes through B and C is tangent to AB. These two circles intersect at a point K other than B. If O is the circumcenter of ABC, prove that $\angle BKO = 90^\circ$.

7 Positive numbers x, y satisfy $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

8 In a group of 12 persons, among any 9 there are 5 which know each other. Prove that there are 6 persons in this group which know each other

– Grade level 11

Day 1

1 Do there exist 19 distinct natural numbers with equal sums of digits, whose sum equals 1999?

2 Each rational point on a real line is assigned an integer. Prove that there is a segment such that the sum of the numbers at its endpoints does not exceed twice the number at its midpoint.

3 A circle touches sides DA, AB, BC, CD of a quadrilateral $ABCD$ at points K, L, M, N, respectively. Let S_1, S_2, S_3, S_4 respectively be the incircles of triangles AKL, BLM, CMN, DNK. The external common tangents distinct from the sides of $ABCD$ are drawn to S_1 and S_2, S_2 and S_3, S_3 and S_4, S_4 and S_1. Prove that these four tangents determine a rhombus.

4 A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

Day 2

5 Four natural numbers are such that the square of the sum of any two of them is divisible by the product of the other two numbers. Prove that at least three of these numbers are equal.

6 Three convex polygons are given on a plane. Prove that there is no line cutting all the polygons if and only if each of the polygons can be separated from the other two by a line.

7 Through vertex A of a tetrahedron $ABCD$ passes a plane tangent to the circumscribed sphere of the tetrahedron. Show that the lines of intersection of the plane with the planes ABC, ABD, ACD, form six equal angles if and only if:

$$AB \cdot CD = AC \cdot BD = AD \cdot BC$$

8 There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts two or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?