All-Russian Olympiad 2001

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– Grade level 9

Day 1

1 The integers from 1 to 999999 are partitioned into two groups: the first group consists of those integers for which the closest perfect square is odd, whereas the second group consists of those for which the closest perfect square is even. In which group is the sum of the elements greater?

2 The two polynomials $(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ take negative values on an interval I of length greater than 2, and nonnegative values outside of I. Prove that there exists $x_0 \in \mathbb{R}$ such that $P(x_0) < Q(x_0)$.

3 A point K is taken inside parallelogram $ABCD$ so that the midpoint of AD is equidistant from K and C, and the midpoint of CD is equidistant form K and A. Let N be the midpoint of BK. Prove that the angles NAK and NCK are equal.

4 Consider a convex 2000-gon, no three of whose diagonals have a common point. Each of its diagonals is colored in one of 999 colors. Prove that there exists a triangle all of whose sides lie on diagonals of the same color. (Vertices of the triangle need not be vertices of the original polygon.)

Day 2

1 Yura put 2001 coins of 1, 2 or 3 kopeykas in a row. It turned out that between any two 1-kopeyka coins there is at least one coin; between any two 2-kopeykas coins there are at least two coins; and between any two 3-kopeykas coins there are at least 3 coins. How many 3-koyepkas coins could Yura put?

2 In a party, there are $2n + 1$ people. It's well known that for every group of n people, there exist a person(out of the group) who knows all them(the n people of the group). Show that there exist a person who knows all the people in the party.

3 Let N be a point on the longest side AC of a triangle ABC. The perpendicular bisectors of AN and NC intersect AB and BC respectively in K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle KBM.

4 Find all odd positive integers $n > 1$ such that if a and b are relatively prime divisors of n, then $a+b-1$ divides n.

– Grade level 10

Day 1

1 The integers from 1 to 999999 are partitioned into two groups: the first group consists of those integers for which the closest perfect square is odd, whereas the second group consists of those for which the closest perfect square is even. In which group is the sum of the elements greater?

2 Let $A_1, A_2, ..., A_{100}$ be subsets of a line, each a union of 100 pairwise disjoint closed segments. Prove that the intersection of all hundred sets is a union of at most 9901 disjoint closed segments.

3 Let the circle ω_1 be internally tangent to another circle ω_2 at N.Take a point K on ω_1 and draw a tangent AB which intersects ω_2 at A and B. Let M be the midpoint of the arc AB which is on the opposite side of N. Prove that, the circumradius of the $\triangle KBM$ doesnt depend on the choice of K.

4 Some towns in a country are connected by twoway roads, so that for any two towns there is a unique path along the roads connecting them. It is known that there is exactly 100 towns which are directly connected to only one town. Prove that we can construct 50 new roads in order to obtain a net in which every two towns will be connected even if one road gets closed.

Day 2

1 The polynomial $P(x) = x^3 + ax^2 + bx + d$ has three distinct real roots. The polynomial $P(Q(x))$, where $Q(x) = x^2 + x + 2001$, has no real roots. Prove that $P(2001) > \frac{1}{64}$.

2 In a magic square $n \times n$ composed from the numbers $1, 2, \cdots, n^2$, the centers of any two squares are joined by a vector going from the smaller number to the bigger one. Prove that the sum of all these vectors is zero. (A magic square is a square matrix such that the sums of entries in all its rows and columns are equal.)

3 Points A_1, B_1, C_1 inside an acute-angled triangle ABC are selected on the altitudes from A, B, C respectively so that the sum of the areas of triangles ABC_1, BCA_1, and CAB_1 is equal to the area of triangle ABC. Prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of triangle ABC.

4 Find all odd positive integers $n > 1$ such that if a and b are relatively prime divisors of n, then $a + b - 1$ divides n.

– Grade level 11

Day 1

1 The total mass of 100 given weights with positive masses equals $2S$. A natural number k is called *middle* if some k of the given weights have the total mass S. Find the maximum possible number of middle numbers.

2 Let the circle ω_1 be internally tangent to another circle ω_2 at N.Take a point K on ω_1 and draw a tangent AB which intersects ω_2 at A and B. Let M be the midpoint of the arc AB which is on the opposite side of N. Prove that, the circumradius of the $\triangle KBM$ doesnt depend on the choice of K.

3 There are two families of convex polygons in the plane. Each family has a pair of disjoint polygons. Any polygon from one family intersects any polygon from the other family. Show that there is a line which intersects all the polygons.

4 Participants to an olympiad worked on n problems. Each problem was worth a positive integer number of points, determined by the jury. A contestant gets 0 points for a wrong answer, and all points for a correct answer to a problem. It turned out after the olympiad that the jury could impose worths of the problems, so as to obtain any (strict) final ranking of the contestants. Find the greatest possible number of contestants.

Day 2

1 Two monic quadratic trinomials $f(x)$ and $g(x)$ take negative values on disjoint intervals. Prove that there exist positive numbers α and β such that $\alpha f(x) + \beta g(x) > 0$ for all real x.

2 Let a, b be 2 distinct positive interger number such that $(a^2 + ab + b^2)|ab(a + b)$. Prove that: $|a - b| > \sqrt[3]{ab}$.

3 The 2001 towns in a country are connected by some roads, at least one road from each town, so that no town is connected by a road to every other city. We call a set D of towns *dominant* if every town not in D is connected by a road to a town in D. Suppose that each dominant set consists of at least k towns. Prove that the country can be partitioned into $2001 - k$ republics in such a way that no two towns in the same republic are connected by a road.

4 A sphere with center on the plane of the face ABC of a tetrahedron $SABC$ passes through A, B and C, and meets the edges SA, SB, SC again at A_1, B_1, C_1, respectively. The planes through A_1, B_1, C_1 tangent to the sphere meet at O. Prove that O is the circumcenter of the tetrahedron $SA_1B_1C_1$.