All-Russian Olympiad 2004

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– Grade level 9

Day 1

1 Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

2 Let $ABCD$ be a circumscribed quadrilateral (i. e. a quadrilateral which has an incircle). The exterior angle bisectors of the angles DAB and ABC intersect each other at K; the exterior angle bisectors of the angles ABC and BCD intersect each other at L; the exterior angle bisectors of the angles BCD and CDA intersect each other at M; the exterior angle bisectors of the angles CDA and DAB intersect each other at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of the triangles ABK, BCL, CDM and DAN, respectively. Show that the quadrilateral $K_1L_1M_1N_1$ is a parallelogram.

3 On a table there are 2004 boxes, and in each box a ball lies. I know that some the balls are white and that the number of white balls is even. In each case I may point to two arbitrary boxes and ask whether in the box contains at least a white ball lies. After which minimum number of questions I can indicate two boxes for sure, in which white balls lie?

4 Let $n > 3$ be a natural number, and let x_1, x_2, ..., x_n be n positive real numbers whose product is 1. Prove the inequality

$$\frac{1}{1+x_1+x_1\cdot x_2}+\frac{1}{1+x_2+x_2\cdot x_3}+...+\frac{1}{1+x_n+x_n\cdot x_1}>1.$$

Day 2

1 Are there such pairwise distinct natural numbers m, n, p, q satisfying $m+n=p+q$ and $\sqrt{m}+\sqrt[3]{n}=\sqrt{p}+\sqrt[3]{q}>2004$?

2 In the cabinet 2004 telephones are located; each two of these telephones are connected by a cable, which is colored in one of four colors. From each color there is one cable at least. Can one always select several telephones in such a way that among their pairwise cable connections exactly 3 different colors occur?

3 The natural numbers from 1 to 100 are arranged on a circle with the characteristic that each number is either larger as their two neighbours or smaller than their two neighbours. A pair of neighbouring numbers is called "good", if you cancel such a pair, the above property remains still valid. What is the smallest possible number of good pairs?

4 Let O be the circumcenter of an acute-angled triangle ABC, let T be the circumcenter of the triangle AOC, and let M be the midpoint of the segment AC. We take a point D on the side AB and a point E on the side BC that satisfy $\angle BDM = \angle BEM = \angle ABC$. Show that the straight lines BT and DE are perpendicular.

– Grade level 10

Day 1

1 Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

2 Let $ABCD$ be a circumscribed quadrilateral (i. e. a quadrilateral which has an incircle). The exterior angle bisectors of the angles DAB and ABC intersect each other at K; the exterior angle bisectors of the angles ABC and BCD intersect each other at L; the exterior angle bisectors of the angles BCD and CDA intersect each other at M; the exterior angle bisectors of the angles CDA and DAB intersect each other at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of the triangles ABK, BCL, CDM and DAN, respectively. Show that the quadrilateral $K_1L_1M_1N_1$ is a parallelogram.

3 Let $ABCD$ be a quadrilateral which is a cyclic quadrilateral and a tangent quadrilateral simultaneously. (By a *tangent quadrilateral*, we mean a quadrilateral that has an incircle.)

Let the incircle of the quadrilateral $ABCD$ touch its sides AB, BC, CD, and DA in the points K, L, M, and N, respectively. The exterior angle bisectors of the angles DAB and ABC intersect each other at a point K'. The exterior angle bisectors of the angles ABC and BCD intersect each other at a point L'. The exterior angle bisectors of the angles BCD and CDA intersect each other at a point M'. The exterior angle bisectors of the angles CDA and DAB intersect each other at a point N'. Prove that the straight lines KK', LL', MM', and NN' are concurrent.

Day 2

1 A sequence of non-negative rational numbers $a(1), a(2), a(3), \ldots$ satisfies $a(m)+a(n) = a(mn)$ for arbitrary natural m and n. Show that not all elements of the sequence can be distinct.

2 A country has 1001 cities, and each two cities are connected by a one-way street. From each city exactly 500 roads begin, and in each city 500 roads end. Now an independent republic splits itself off the country, which contains 668 of the 1001 cities. Prove that one can reach every other city of the republic from each city of this republic without being forced to leave the republic.

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3 A triangle T is contained inside a point-symmetrical polygon M. The triangle T' is the mirror image of the triangle T with the reflection at one point P, which inside the triangle T lies. Prove that at least one of the vertices of the triangle T' lies in inside or on the boundary of the polygon M.

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4 Is there a natural number $n > 10^{1000}$ which is not divisible by 10 and which satisfies: in its decimal representation one can exchange two distinct non-zero digits such that the set of prime divisors does not change.

– Grade level 11

Day 1

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1 Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have pairwise different colors.

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2 Let $I(A)$ and $I(B)$ be the centers of the excircles of a triangle ABC, which touches the sides BC and CA in its interior. Furthermore let P a point on the circumcircle ω of the triangle ABC. Show that the center of the segment which connects the circumcenters of the triangles $I(A)CP$ and $I(B)CP$ coincides with the center of the circle ω.

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3 The polynomials $P(x)$ and $Q(x)$ are given. It is known that for a certain polynomial $R(x,y)$ the identity $P(x)-P(y)=R(x,y)(Q(x)-Q(y))$ applies. Prove that there is a polynomial $S(x)$ so that $P(x)=S(Q(x)) \quad \forall x$.

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4 A rectangular array has 9 rows and 2004 columns. In the 9 * 2004 cells of the table we place the numbers from 1 to 2004, each 9 times. And we do this in such a way that two numbers, which stand in exactly the same column in and differ around at most by 3. Find the smallest possible sum of all numbers in the first row.

Day 2

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1 Let $M=\{x_1 ..., x_{30}\}$ a set which consists of 30 distinct positive numbers, let A_n, $1\leq n\leq 30$, the sum of all possible products with n elements each of the set M. Prove if $A_{15}>A_{10}$, then $A_1>1$.

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2 Prove that there is no finite set which contains more than $2N$, with $N > 3$, pairwise non-collinear vectors of the plane, and to which the following two characteristics apply:

1) for N arbitrary vectors from this set there are always further $N-1$ vectors from this set so that the sum of these is $2N-1$ vectors is equal to the zero-vector;

2) for N arbitrary vectors from this set there are always further N vectors from this set so that the sum of these is $2N$ vectors is equal to the zero-vector.

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3 In a country there are several cities; some of these cities are connected by airlines, so that an airline connects exactly two cities in each case and both flight directions are possible. Each airline belongs to one of k flight companies; two airlines of the same flight company have always a common final point. Show that one can partition all cities in $k+2$ groups in such a way that two cities from exactly the same group are never connected by an airline with each other.

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4 A parallelepiped is cut by a plane along a 6-gon. Supposed this 6-gon can be put into a certain rectangle π (which means one can put the rectangle π on the parallelepiped's plane such that the 6-gon is completely covered by the rectangle). Show that one also can put one of the parallelepiped' faces into the rectangle π.
