International Olympiad of Metropolises

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Day 1 .

1 Let $ABCD$ be a parallelogram in which angle at B is obtuse and $AD > AB$. Points K and L on AC such that $\angle ADL = \angle KBA$(the points A, K, C, L are all different, with K between A and L). The line BK intersects the circumcircle ω of ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF||AC$.

2 In a country there are two-way non-stopflights between some pairs of cities. Any city can be reached from any other by a sequence of at most 100 flights. Moreover, any city can be reached from any other by a sequence of an even number of flights. What is the smallest d for which one can always claim that any city can be reached from any other by a sequence of an even number of flights not exceeding d?

3 Let Q be a quadriatic polynomial having two different real zeros. Prove that there is a non-constant monic polynomial P such that all coefficients of the polynomial $Q(P(x))$ except the leading one are (by absolute value) less than 0.001.

Day 2 .

4 Find the largest positive integer N for which one can choose N distinct numbers from the set $1, 2, 3, ..., 100$ such that neither the sum nor the product of any two different chosen numbers is divisible by 100.

Proposed by Mikhail Evdokimov

5 Let x and y be positive integers such that $[x+2, y+2] - [x+1, y+1] = [x+1, y+1] - [x, y]$.Prove that one of the two numbers x and y divide the other.

(Here $[a, b]$ denote the least common multiple of a and b).

Proposed by Dusan Djukic.

6 et $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumcribed. Denote by $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$ and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF and EFA, respecitively. Let l_{AB}, be the external of ω_A and ω_B; lines l_{BC}, l_{CD}, l_{DE}, l_{EF}, l_{FA} are analoguosly defined. Let A_1 be the intersection point of the lines l_{FA} and l_{AB}, B_1, C_1, D_1, E_1, F_1 are analogously defined.
Prove that A_1D_1, B_1E_1, C_1F_1 are concurrent.