All-Russian Olympiad 2006

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–	Grade level 9
1	Given a 15×15 chessboard. We draw a closed broken line without self-intersections such that every edge of the broken line is a segment joining the centers of two adjacent cells of the chessboard. If this broken line is symmetric with respect to a diagonal of the chessboard, then show that the length of the broken line is ≤ 200.
2	Show that there exist four integers a, b, c, d whose absolute values are all > 1000000 and which satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{abcd}$.
3	Given a circle and 2006 points lying on this circle. Albatross colors these 2006 points in 17 colors. After that, Frankinfueter joins some of the points by chords such that the endpoints of each chord have the same color and two different chords have no common points (not even a common endpoint). Hereby, Frankinfueter intends to draw as many chords as possible, while Albatross is trying to hinder him as much as he can. What is the maximal number of chords Frankinfueter will always be able to draw?
4	Given a triangle ABC. Let a circle ω touch the circumcircle of triangle ABC at the point A, intersect the side AB at a point K, and intersect the side BC. Let CL be a tangent to the circle ω, where the point L lies on ω and the segment KL intersects the side BC at a point T. Show that the segment BT has the same length as the tangent from the point B to the circle ω.
5	Let a_1, a_2, ..., a_{10} be positive integers such that $a_1 < a_2 < ... < a_{10}$. For every k, denote by b_k the greatest divisor of a_k such that $b_k < a_k$. Assume that $b_1 > b_2 > ... > b_{10}$. Show that $a_{10} > 500$.
6	Let P, Q, R be points on the sides AB, BC, CA of a triangle ABC such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle ABC at the points C and A intersect the lines RQ and RP at the points X and Y, respectively. Prove that $RX = RY$.
7	A 100×100 chessboard is cut into dominoes (1×2 rectangles). Two persons play the following game: At each turn, a player glues together two adjacent cells (which were formerly separated by a cut-edge). A player loses if, after his turn, the 100×100 chessboard becomes connected, i. e. between any two cells there exists a way which doesn't intersect any cut-edge. Which player has a winning strategy - the starting player or his opponent?
8	Given a quadratic trinomial $f(x) = x^2 + ax + b$. Assume that the equation $f(f(x)) = 0$ has four

different real solutions, and that the sum of two of these solutions is -1. Prove that $b \leq -\frac{1}{4}$.

– Grade level 10

1 Given a 15×15 chessboard. We draw a closed broken line without self-intersections such that every edge of the broken line is a segment joining the centers of two adjacent cells of the chessboard. If this broken line is symmetric with respect to a diagonal of the chessboard, then show that the length of the broken line is ≤ 200.

2 If an integer $a > 1$ is given such that $(a-1)^3 + a^3 + (a+1)^3$ is the cube of an integer, then show that $4 \mid a$.

3 Given a circle and 2006 points lying on this circle. Albatross colors these 2006 points in 17 colors. After that, Frankinfueter joins some of the points by chords such that the endpoints of each chord have the same color and two different chords have no common points (not even a common endpoint). Hereby, Frankinfueter intends to draw as many chords as possible, while Albatross is trying to hinder him as much as he can. What is the maximal number of chords Frankinfueter will always be able to draw?

4 Consider an isosceles triangle ABC with $AB = AC$, and a circle ω which is tangent to the sides AB and AC of this triangle and intersects the side BC at the points K and L. The segment AK intersects the circle ω at a point M (apart from K). Let P and Q be the reflections of the point K in the points B and C, respectively. Show that the circumcircle of triangle PMQ is tangent to the circle ω.

5 Let a_1, a_2, ..., a_{10} be positive integers such that $a_1 < a_2 < ... < a_{10}$. For every k, denote by b_k the greatest divisor of a_k such that $b_k < a_k$. Assume that $b_1 > b_2 > ... > b_{10}$. Show that $a_{10} > 500$.

6 Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that $KL \parallel AC$. Show that the incenters of triangles ABK and CBL are equidistant from the midpoint of the arc ABC of the circumcircle of triangle ABC.

7 Given a quadratic trinomial $f(x) = x^2 + ax + b$. Assume that the equation $f(f(x)) = 0$ has four different real solutions, and that the sum of two of these solutions is -1. Prove that $b \leq -\frac{1}{4}$.

8 A 3000×3000 square is tiled by dominoes (i. e. 1×2 rectangles) in an arbitrary way. Show that one can color the dominoes in three colors such that the number of the dominoes of each color is the same, and each domino d has at most two neighbours of the same color as d. (Two dominoes are said to be *neighbours* if a cell of one domino has a common edge with a cell of the other one.)

– Grade level 11

1 Prove that $\sin\sqrt{x} < \sqrt{\sin x}$ for every real x such that $0 < x < \frac{\pi}{2}$.

2 The sum and the product of two purely periodic decimal fractions a and b are purely periodic decimal fractions of period length T. Show that the lengths of the periods of the fractions a and b are not greater than T.
Note. A *purely periodic decimal fraction* is a periodic decimal fraction without a non-periodic starting part.

3 On a 49×69 rectangle formed by a grid of lattice squares, all $50 \cdot 70$ lattice points are colored blue. Two persons play the following game: In each step, a player colors two blue points red, and draws a segment between these two points. (Different segments can intersect in their interior.) Segments are drawn this way until all formerly blue points are colored red. At this moment, the first player directs all segments drawn - i. e., he takes every segment AB, and replaces it either by the vector $\overrightarrow{AB}$, or by the vector $\overrightarrow{BA}$. If the first player succeeds to direct all the segments drawn in such a way that the sum of the resulting vectors is $\overrightarrow{0}$, then he wins; else, the second player wins.
Which player has a winning strategy?

4 Given a triangle ABC. The angle bisectors of the angles ABC and BCA intersect the sides CA and AB at the points B_1 and C_1, and intersect each other at the point I. The line B_1C_1 intersects the circumcircle of triangle ABC at the points M and N. Prove that the circumradius of triangle MIN is twice as long as the circumradius of triangle ABC.

5 Two sequences of positive reals, (x_n) and (y_n), satisfy the relations $x_{n+2} = x_n + x_{n+1}^2$ and $y_{n+2} = y_n^2 + y_{n+1}$ for all natural numbers n. Prove that, if the numbers x_1, x_2, y_1, y_2 are all greater than 1, then there exists a natural number k such that $x_k > y_k$.

6 Consider a tetrahedron $SABC$. The incircle of the triangle ABC has the center I and touches its sides BC, CA, AB at the points E, F, D, respectively. Let A', B', C' be the points on the segments SA, SB, SC such that $AA' = AD$, $BB' = BE$, $CC' = CF$, and let S' be the point diametrically opposite to the point S on the circumsphere of the tetrahedron $SABC$. Assume that the line SI is an altitude of the tetrahedron $SABC$. Show that $S'A' = S'B' = S'C'$.

7 Assume that the polynomial $(x+1)^n - 1$ is divisible by some polynomial $P(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + ... + c_1x + c_0$, whose degree k is even and whose coefficients c_{k-1}, c_{k-2}, ..., c_1, c_0 all are odd integers. Show that $k+1 \mid n$.

8 At a tourist camp, each person has at least 50 and at most 100 friends among the other persons at the camp. Show that one can hand out a t-shirt to every person such that the t-shirts have (at most) 1331 different colors, and any person has 20 friends whose t-shirts all have pairwisely different colors.