All-Russian Olympiad 2009

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by April, Namdung

–	Grade level 9
1	The denominators of two irreducible fractions are 600 and 700. Find the minimum value of the denominator of their sum (written as an irreducible fraction).
2	Let be given a triangle ABC and its internal angle bisector BD $(D \in BC)$. The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.
3	Given are positive integers $n > 1$ and a so that $a > n^2$, and among the integers $a+1, a+2, \ldots, a+n$ one can find a multiple of each of the numbers $n^2+1, n^2+2, \ldots, n^2+n$. Prove that $a > n^4 - n^3$.
4	There are n cups arranged on the circle. Under one of cups is hiden a coin. For every move, it is allowed to choose 4 cups and verify if the coin lies under these cups. After that, the cups are returned into its former places and the coin moves to one of two neigbor cups. What is the minimal number of moves we need in order to eventually find where the coin is?
5	Let a, b, c be three real numbers satisfying that $\begin{cases} (a+b)(b+c)(c+a) = abc \\ (a^3+b^3)(b^3+c^3)(c^3+a^3) = a^3b^3c^3 \end{cases}$ Prove that $abc = 0$.
6	Can be colored the positive integers with 2009 colors if we know that each color paints infinitive integers and that we can not find three numbers colored by three different colors for which the product of two numbers equal to the third one?
7	We call any eight squares in a diagonal of a chessboard as a fence. The rook is moved on the chessboard in such way that he stands neither on each square over one time nor on the squares of the fences (the squares which the rook passes is not considered ones it has stood on). Then what is the maximum number of times which the rook jumped over the fence?
8	Triangles ABC and $A_1B_1C_1$ have the same area. Using compass and ruler, can we always construct triangle $A_2B_2C_2$ equal to triangle $A_1B_1C_1$ so that the lines AA_2, BB_2, and CC_2 are parallel?

– Grade level 10

1 Find all value of n for which there are nonzero real numbers a, b, c, d such that after expanding and collecting similar terms, the polynomial $(ax+b)^{100} - (cx+d)^{100}$ has exactly n nonzero coefficients.

2 Let be given a triangle ABC and its internal angle bisector BD $(D \in BC)$. The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

3 How many times changes the sign of the function

$$f(x) = \cos x \cos \frac{x}{2} \cos \frac{x}{3} \cdots \cos \frac{x}{2009}$$

at the interval $[0, \frac{2009\pi}{2}]$?

4 On a circle there are 2009 nonnegative integers not greater than 100. If two numbers sit next to each other, we can increase both of them by 1. We can do this at most k times. What is the minimum k so that we can make all the numbers on the circle equal?

5 Given strictly increasing sequence $a_1 < a_2 < \dots$ of positive integers such that each its term a_k is divisible either by 1005 or 1006, but neither term is divisible by 97. Find the least possible value of maximal difference of consecutive terms $a_{i+1} - a_i$.

6 Given a finite tree T and isomorphism $f : T \to T$. Prove that either there exist a vertex a such that $f(a) = a$ or there exist two neighbor vertices a, b such that $f(a) = b$, $f(b) = a$.

7 The incircle (I) of a given scalene triangle ABC touches its sides BC, CA, AB at A_1, B_1, C_1, respectively. Denote ω_B, ω_C the incircles of quadrilaterals BA_1IC_1 and CA_1IB_1, respectively. Prove that the internal common tangent of ω_B and ω_C different from IA_1 passes through A.

8 Let x, y be two integers with $2 \le x, y \le 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.

– Grade level 11

1 In a country, there are some cities linked together by roads. The roads just meet each other inside the cities. In each city, there is a board which showing the shortest length of the road originating in that city and going through all other cities (the way can go through some cities more than one times and is not necessary to turn back to the originated city). Prove that 2 random numbers in the boards can't be greater or lesser than 1.5 times than each other.

2 Consider the sequence of numbers (a_n) $(n = 1, 2, \dots)$ defined as follows: $a_1 \in (1, 2)$, $a_{k+1} = a_k + \frac{k}{a_k}$ $(k = 1, 2, \dots)$. Prove that there exists at most one pair of distinct positive integers (i, j) such that $a_i + a_j$ is an integer.

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3 Let $ABCD$ be a triangular pyramid such that no face of the pyramid is a right triangle and the orthocenters of triangles ABC, ABD, and ACD are collinear. Prove that the center of the sphere circumscribed to the pyramid lies on the plane passing through the midpoints of AB, AC and AD.

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4 Given a set M of points (x, y) with integral coordinates satisfying $x^2 + y^2 \le 10^{10}$. Two players play a game. One of them marks a point on his first move. After this, on each move the moving player marks a point, which is not yet marked and joins it with the previous marked point. Players are not allowed to mark a point symmetrical to the one just chosen. So, they draw a broken line. The requirement is that lengths of edges of this broken line must strictly increase. The player, which can not make a move, loses. Who have a winning strategy?

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5 Prove that

$$\log_a b + \log_b c + \log_c a \le \log_b a + \log_c b + \log_a c$$

for all $1 < a \le b \le c$.

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6 There are k rooks on a 10×10 chessboard. We mark all the squares that at least one rook can capture (we consider the square where the rook stands as captured by the rook). What is the maximum value of k so that the following holds for some arrangement of k rooks: after removing any rook from the chessboard, there is at least one marked square not captured by any of the remaining rooks.

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7 Let be given a parallelogram $ABCD$ and two points A_1, C_1 on its sides AB, BC, respectively. Lines AC_1 and CA_1 meet at P. Assume that the circumcircles of triangles AA_1P and CC_1P intersect at the second point Q inside triangle ACD. Prove that $\angle PDA = \angle QBA$.

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8 Let x, y be two integers with $2 \le x, y \le 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.
