All-Russian Olympiad 2011

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by 3333, sartt, Bugi

– Grade 9

Day 1

1 A quadratic trinomial $P(x)$ with the x^2 coefficient of one is such, that $P(x)$ and $P(P(P(x)))$ share a root. Prove that $P(0)*P(1)=0$.

2 Given is an acute angled triangle ABC. A circle going through B and the triangle's circumcenter, O, intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on line AC.

3 A convex 2011-gon is drawn on the board. Peter keeps drawing its diagonals in such a way, that each newly drawn diagonal intersected no more than one of the already drawn diagonals. What is the greatest number of diagonals that Peter can draw?

4 Do there exist any three relatively prime natural numbers so that the square of each of them is divisible by the sum of the two remaining numbers?

Day 2

1 For some 2011 natural numbers, all the $\frac{2010\cdot 2011}{2}$ possible sums were written out on a board. Could it have happened that exactly one third of the written numbers were divisible by three and also exactly one third of them give a remainder of one when divided by three?

2 In the notebooks of Peter and Nick, two numbers are written. Initially, these two numbers are 1 and 2 for Peter and 3 and 4 for Nick. Once a minute, Peter writes a quadratic trinomial $f(x)$, the roots of which are the two numbers in his notebook, while Nick writes a quadratic trinomial $g(x)$ the roots of which are the numbers in *his* notebook. If the equation $f(x)=g(x)$ has two distinct roots, one of the two boys replaces the numbers in his notebook by those two roots. Otherwise, nothing happens. If Peter once made one of his numbers 5, what did the other one of his numbers become?

3 Let ABC be an equilateral triangle. A point T is chosen on AC and on arcs AB and BC of the circumcircle of ABC, M and N are chosen respectively, so that MT is parallel to BC and NT is parallel to AB. Segments AN and MT intersect at point X, while CM and NT intersect in point Y. Prove that the perimeters of the polygons $AXYC$ and $XMBNY$ are the same.

4 There are some counters in some cells of 100×100 board. Call a cell *nice* if there are an even number of counters in adjacent cells. Can exactly one cell be *nice*?

K. Knop

– Grade 10

Day 1

1 In every cell of a table with n rows and ten columns, a digit is written. It is known that for every row A and any two columns, you can always find a row that has different digits from A only when it intersects with two columns. Prove that $n \geq 512$.

2 Nine quadratics, $x^2 + a_1x + b_1, x^2 + a_2x + b_2, ..., x^2 + a_9x + b_9$ are written on the board. The sequences $a_1, a_2, ..., a_9$ and $b_1, b_2, ..., b_9$ are arithmetic. The sum of all nine quadratics has at least one real root. What is the the greatest possible number of original quadratics that can have no real roots?

3 The graph G is not 3-coloured. Prove that G can be divided into two graphs M and N such that M is not 2-coloured and N is not 1-coloured.

V. Dolnikov

4 Perimeter of triangle ABC is 4. Point X is marked at ray AB and point Y is marked at ray AC such that AX=AY=1. BC intersects XY at point M. Prove that perimeter of one of triangles ABM or ACM is 2.
(V. Shmarov).

Day 2

1 Given are 10 distinct real numbers. Kyle wrote down the square of the difference for each pair of those numbers in his notebook, while Peter wrote in his notebook the absolute value of the differences of the squares of these numbers. Is it possible for the two boys to have the same set of 45 numbers in their notebooks?

2 Given is an acute triangle ABC. Its heights BB_1 and CC_1 are extended past points B_1 and C_1. On these extensions, points P and Q are chosen, such that angle PAQ is right. Let AF be a height of triangle APQ. Prove that angle BFC is a right angle.

3 For positive integers $a > b > 1$, define

$$x_n = \frac{a^n - 1}{b^n - 1}$$

Find the least d such that for any a, b, the sequence x_n does not contain d consecutive prime numbers.

V. Senderov

4 A 2010×2010 board is divided into corner-shaped figures of three cells. Prove that it is possible to mark one cell in each figure such that each row and each column will have the same number of marked cells.

I. Bogdanov & O. Podlipsky

– Grade 11

Day 1

1 Two natural numbers d and d', where $d' > d$, are both divisors of n. Prove that $d' > d + \frac{d^2}{n}$.

2 On side BC of parallelogram $ABCD$ (A is acute) lies point T so that triangle ATD is an acute triangle. Let O_1, O_2, and O_3 be the circumcenters of triangles ABT, DAT, and CDT respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line AD.

3 There are 999 scientists. Every 2 scientists are both interested in exactly 1 topic and for each topic there are exactly 3 scientists that are interested in that topic. Prove that it is possible to choose 250 topics such that every scientist is interested in at most 1 theme.

A. Magazinov

4 Ten cars are moving at the road. There are some cities at the road. Each car is moving with some constant speed through cities and with some different constant speed outside the cities (different cars may move with different speed). There are 2011 points at the road. Cars don't overtake at the points. Prove that there are 2 points such that cars pass through these points in the same order.

S. Berlov

Day 2

1 Given are two distinct monic cubics $F(x)$ and $G(x)$. All roots of the equations $F(x) = 0$, $G(x) = 0$ and $F(x) = G(x)$ are written down. There are eight numbers written. Prove that the greatest of them and the least of them cannot be both roots of the polynomial $F(x)$.

2 There are more than n^2 stones on the table. Peter and Vasya play a game, Peter starts. Each turn, a player can take any prime number less than n stones, or any multiple of n stones, or 1 stone. Prove that Peter always can take the last stone (regardless of Vasya's strategy).

S Berlov

3 Let $P(a)$ be the largest prime positive divisor of a^2+1. Prove that exist infinitely many positive integers a, b, c such that $P(a)=P(b)=P(c)$.

A. Golovanov

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4 Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, let M be the midpoint of AC and let I_1, I_2 be the incentres of triangles ABM and CBM. Prove that points I_1, I_2, B, N lie on a circle.

M. Kungojin
