Greece National Olympiad 1996

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1 Let a_n be a sequence of positive numbers such that:
i) $\dfrac{a_{n+2}}{a_n} = \dfrac{1}{4}$, for every $n \in \mathbb{N}^\star$
ii) $\dfrac{a_{k+1}}{a_k} + \dfrac{a_{n+1}}{a_n} = 1$, for every $k, n \in \mathbb{N}^\star$ with $|k-n| \neq 1$.
(a) Prove that (a_n) is a geometric progression.
(n) Prove that exists $t > 0$, such that $\sqrt{a_{n+1}} \leq \dfrac{1}{2}a_n + t$

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2 Let ABC be an acute triangle, AD, BE, CZ its altitudes and H its orthocenter. Let $AI, A\Theta$ be the internal and external bisectors of angle A. Let M, N be the midpoints of BC, AH, respectively. Prove that:
(a) MN is perpendicular to EZ
(b) if MN cuts the segments $AI, A\Theta$ at the points K, L, then $KL = AH$

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3 Prove that among 81 natural numbers whose prime divisors are in the set $\{2, 3, 5\}$ there exist four numbers whose product is the fourth power of an integer.

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4 Find the number of functions $f : \{1, 2, ..., n\} \to \{1995, 1996\}$ such that $f(1) + f(2) + ... + f(1996)$ is odd.
