Greece National Olympiad 2001

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by socrates

1 A triangle ABC is inscribed in a circle of radius R. Let BD and CE be the bisectors of the angles B and C respectively and let the line DE meet the arc AB not containing C at point K. Let A_1, B_1, C_1 be the feet of perpendiculars from K to BC, AC, AB, and x, y be the distances from D and E to BC, respectively.

(a) Express the lengths of KA_1, KB_1, KC_1 in terms of x, y and the ratio $l = KD/ED$.

(b) Prove that $\frac{1}{KB} = \frac{1}{KA} + \frac{1}{KC}$.

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2 Prove that there are no positive integers a, b such that $(15a+b)(a+15b)$ is a power of 3.

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3 A function $f : \mathbb{N}_0 \to \mathbb{R}$ satises $f(1) = 3$ and

$$f(m+n) + f(m-n) - m + n - 1 = \frac{f(2m) + f(2n)}{2},$$

for any non-negative integers m and n with $m \geq n$. Find all such functions f.

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4 The numbers 1 to 500 are written on a board. Two pupils A and B play the following game: A player in turn deletes one of the numbers from the board. The game is over when only two numbers remain. Player B wins if the sum of the two remaining numbers is divisible by 3, otherwise A wins. If A plays rst, show that B has a winning strategy.
