Greece National Olympiad 2003

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by socrates

1 If a, b, c, d are positive numbers satisfying $a^3 + b^3 + 3ab = c + d = 1$, prove that

$$\left(a+\frac{1}{a}\right)^3+\left(b+\frac{1}{b}\right)^3+\left(c+\frac{1}{c}\right)^3+\left(d+\frac{1}{d}\right)^3\geq 40.$$

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2 Find all real solutions of the system

$$\begin{cases} x^2+y^2-z(x+y)=2, \\ y^2+z^2-x(y+z)=4, \\ z^2+x^2-y(z+x)=8. \end{cases}$$

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3 Given are a circle $\mathcal{C}$ with center K and radius r, point A on the circle and point R in its exterior. Consider a variable line e through R that intersects the circle at two points B and C. Let H be the orthocenter of triangle ABC.

Show that there is a unique point T in the plane of circle $\mathcal{C}$ such that the sum $HA^2 + HT^2$ remains constant (as e varies.)

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4 On the set Σ of points of the plane Π we dene the operation $*$ which maps each pair (X, Y) of points in Σ to the point $Z = X * Y$ that is symmetric to X with respect to Y. Consider a square $ABCD$ in Π. Is it possible, using the points A, B, C and applying the operation $*$ nitely many times, to construct the point D?
