Greece National Olympiad 2005

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–	Seniors
–	February 12th
1	Find the polynomial $P(x)$ with real coefficients such that $P(2) = 12$ and $P(x^2) = x^2(x^2+1)P(x)$ for each $x \in \mathbb{R}$.
2	The sequence (a_n) is defined by $a_1 = 1$ and $a_n = a_{n-1} + \frac{1}{n^3}$ for $n > 1$. (a) Prove that $a_n < \frac{5}{4}$ for all n. (b) Given $\epsilon > 0$, find the smallest natural number n_0 such that $\mid a_{n+1} - a_n \mid < \epsilon$ for all $n > n_0$.
3	We know that k is a positive integer and the equation $$x^3 + y^3 - 2y(x^2 - xy + y^2) = k^2(x - y) \quad (1)$$ has one solution (x_0, y_0) with $x_0, y_0 \in \mathbb{Z} - \{0\}$ and $x_0 \neq y_0$. Prove that i) the equation (1) has a finite number of solutions (x, y) with $x, y \in \mathbb{Z}$ and $x \neq y$; ii) it is possible to find 11 addition different solutions (X, Y) of the equation (1) with $X, Y \in \mathbb{Z} - \{0\}$ and $X \neq Y$ where X, Y are functions of x_0, y_0.
4	Let OX_1, OX_2 be rays in the interior of a convex angle XOY such that $\angle XOX_1 = \angle YOY_1 < \frac{1}{3}\angle XOY$. Points K on OX_1 and L on OY_1 are fixed so that $OK = OL$, and points A, B are vary on rays $(OX, (OY$ respectively such that the area of the pentagon $OAKLB$ remains constant. Prove that the circumcircle of the triangle OAB passes from a fixed point, other than O.