Greece National Olympiad 2009

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by socrates,

1 Find all positive integers n such that the number

$$A=\sqrt{\frac{9n-1}{n+7}}$$

is rational.

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2 Consider a triangle ABC with circumcenter O and let A_1, B_1, C_1 be the midpoints of the sides BC, AC, AB, respectively.
Points A_2, B_2, C_2 are defined as $\overrightarrow{OA_2}=\lambda\cdot\overrightarrow{OA_1},\ \overrightarrow{OB_2}=\lambda\cdot\overrightarrow{OB_1},\ \overrightarrow{OC_2}=\lambda\cdot\overrightarrow{OC_1}$, where $\lambda>0$.
Prove that lines AA_2, BB_2, CC_2 are concurrent.

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3 Let x, y, z be nonnegative real numbers such that $x+y+z=2$. Prove that $x^2y^2+y^2z^2+z^2x^2+xyz\leq 1$. When does the equality occur?

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4 Consider pairwise distinct complex numbers $z_1, z_2, z_3, z_4, z_5, z_6$ whose images $A_1, A_2, A_3, A_4, A_5, A_6$ respectively are succesive points on the circle centered at $O(0,0)$ and having radius $r>0$.
If w is a root of the equation $z^2+z+1=0$ and the next equalities hold

$$z_1w^2+z_3w+z_5=0 z_2w^2+z_4w+z_6=0$$

prove that

a) Triangle $A_1A_3A_5$ is equilateral
b)

$$|z_1-z_2|+|z_2-z_3|+|z_3-z_4|+|z_4-z_5|+z_5-z_6|+|z_6-z_1|=3|z_1-z_4|=3|z_2-z_5|=3|z_3-z_6|.$$
